Relativistic wide-angle galaxy Power Spectrum and Bispectrum on the light-cone: All-sky analysis

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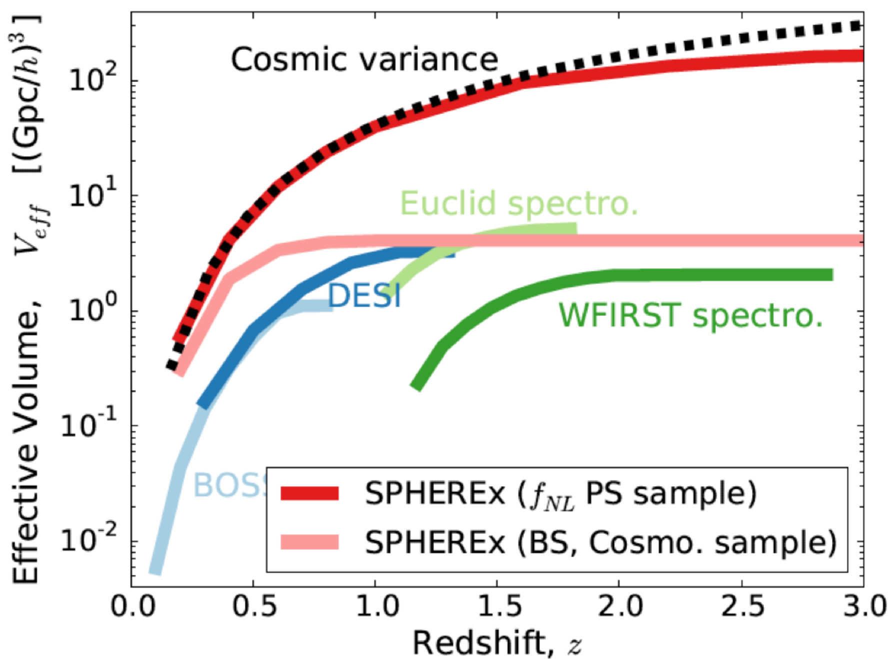
Large-scale measurements
vs
Fourier analysis

- A traditional approach to analyzing galaxy number density fields is to utilize the Power Spectrum (or Bispectrum) of its Fourier components.

- Power Spectrum analysis merits its intuitively simple interpretation of the measurements in connection with the underlying matter distribution, and it assumes that the density fields are defined in a cubic volume.

Large-scale measurements
vs
Spherical Fourier analysis

As the recent and forthcoming galaxy surveys cover a progressively larger fraction of the sky, the validity of the flat-sky approximation and the power Spectrum and Bispectrum analysis become questionable.



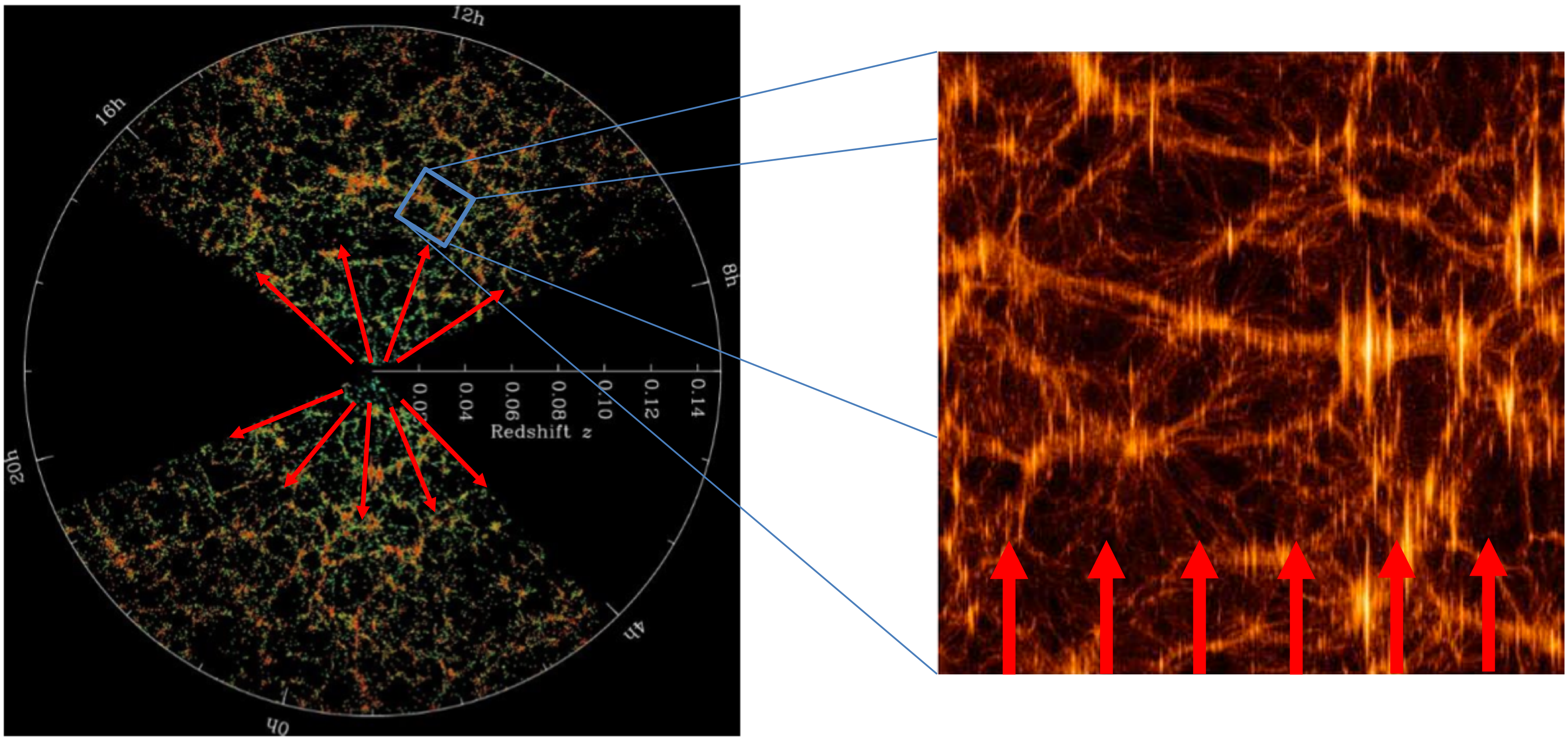
(Doré at al 2014)

Large-scale measurements *vs* Spherical Fourier analysis

- Spherical Fourier analysis expresses the observed galaxy fluctuation in terms of the spherical harmonics and spherical Bessel functions that are angular and radial eigenfunctions of the Helmholtz equation.

- This analysis provides a natural orthogonal basis for all-sky analysis of the large-scale mode measurements.

Large-scale measurements
vs
Spherical Fourier analysis



The Spherical-Bessel formalism

- The “Spherical-Bessel” formalism was first proposed in redshift space by Heavens & Taylor 1995 (see also Peebles 1973, Binney & Quinn 1991, Fisher et al. 1994, Tadros et al. 2000) for the two-point correlation. function

- Then for the three-point statistics in real space, see Verde, Heavens & Matarrese 2000.

- The same formalism (for the power-spectrum) has been applied to real data in Percival et al. 2004 (see also Tegmark et al. 2002) and extended to include GR (projections) effects in Yoo & Desjacques 2013.

- Compared to the traditional weak lensing, this spherical Fourier analysis is known as the 3D weak lensing (see Castro et al. 2005), and it is shown in Kitching et al. 2012 that the 2D tomography in weak lensing is just the 3D weak lensing with the Limber approximation.

Outline:

- Briefly review of the full general relativistic description:
  - of the spherical galaxy Power Spectrum
    (Yoo & Desjacques, arXiv:1301.4501)
  - of the spherical galaxy Bispectrum
    (DB+ arXiv:1705.09306)

see also Yoo & Desjacques(2013)

Spherical Fourier space

Let us consider a complete radial and angular basis $|klm\rangle$ in a spherical Fourier space

Its representation in configuration space is

$$\langle \mathbf{x}|klm\rangle \equiv \sqrt{\frac{2}{\pi}}\, k\, j_l(kr)\, Y_{lm}(\hat{\mathbf{x}})$$

where

$r = |\mathbf{x}|$

$\hat{\mathbf{x}} = (\theta, \phi)$ is the radial position

$j_l(kr)$ is the unit directional vector of $\mathbf{x}$

$Y_{lm}(\hat{\mathbf{x}})$ is the spherical Bessel function

is the spherical harmonics function

see also Yoo & Desjacques(2013)

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$|klm\rangle$ is orthonormal:

$$\langle k'l'm'|klm\rangle \;=\; \int d^3\mathbf{x}\, \langle k'l'm'|\mathbf{x}\rangle\langle \mathbf{x}|klm\rangle = \; \delta^D(k-k')\, \delta_{ll'}\delta_{mm'}$$

see also Yoo & Desjacques (2013)

Spherical Fourier space

Spherical Fourier decomposition of δ

$$\delta(\mathbf{x}) \;=\; \langle \mathbf{x}|\delta\rangle = \int_0^\infty dk \sum_{lm} \langle \mathbf{x}|klm\rangle\langle klm|\delta\rangle$$

$$= \int_0^\infty dk \sum_{lm} \sqrt{\frac{2}{\pi}}\, k\, j_l(kr)\, Y_{lm}(\hat{\mathbf{x}})\, \delta_{lm}(k)$$

then

$$\delta_{lm}(k) \equiv \langle klm|\delta\rangle = \frac{i^l k}{(2\pi)^{3/2}} \int d^2\hat{\mathbf{k}}\, Y^*_{lm}(\hat{\mathbf{k}})\, \delta(\mathbf{k})$$

see also Yoo & Desjacques(2013)

Spherical Power Spectrum: (in real space)

$$\langle \delta_{lm}(k)\, \delta^*_{l'm'}(k') \rangle \equiv \delta_{ll'} \delta_{mm'} \mathcal{S}_l(k,k')$$

$$= \frac{i^l(-i)^{l'}kk'}{(2\pi)^3} \int d^2\hat{\mathbf{k}}\, d^2\hat{\mathbf{k}}'\; Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}') \langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle$$

For a rotationally and translationally invariant Power Spectrum

$$\langle \delta(\mathbf{k}) \delta^*(\mathbf{k}') \rangle \equiv P(\mathbf{k},\mathbf{k}') = (2\pi)^3 \delta^D(\mathbf{k}-\mathbf{k}') P(k)$$

the spherical Power Spectrum is

$$\mathcal{S}_l(k,k') = \delta^D(k-k') \mathcal{S}_l(k) = \delta^D(k-k') P(k)$$

see also Yoo & Desjacques (2013)

Spherical Power Spectrum: (in real space)

Other useful relations:

$$\delta_{lm}(k) = \langle klm|\delta\rangle = \int \mathrm{d}^3\mathbf{x}\ \langle klm|\mathbf{x}\rangle\langle\mathbf{x}|\delta\rangle = \int \mathrm{d}^3\mathbf{x}\sqrt{\frac{2}{\pi}}\, k j_l(kr)\ Y^*_{lm}(\hat{\mathbf{x}})\delta(\mathbf{x})$$

then

$$\mathcal{S}_l(k,k') \ = \ \frac{2kk'}{\pi}\int d^3\mathbf{x}_1 \int d^3\mathbf{x}_2\, Y^*_{lm}(\hat{\mathbf{x}}_1)Y_{lm}(\hat{\mathbf{x}}_2)\, j_l(kr_1)j_l(k'r_2)\ \langle\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\rangle$$

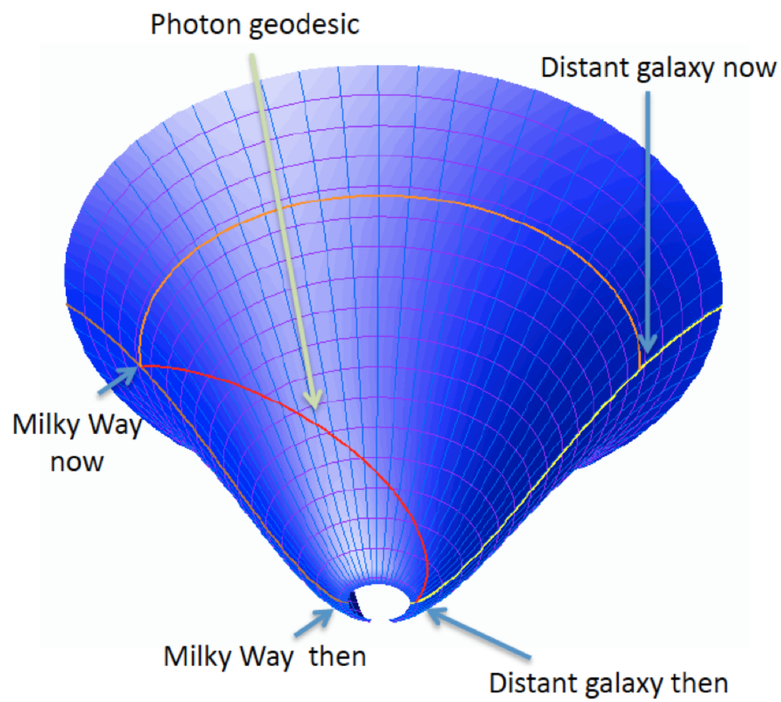
In case that time evolution is related to radial coordinates

⬇

Then, it is more natural these relations for computing spherical Fourier modes and their spherical Power Spectrum.

GALAXY CLUSTERING IN GENERAL RELATIVITY

- The standard approach to modeling galaxy clustering is based on the "Newtonian framework"
- It naturally breaks down on large scales, where relativistic (projection) effects becomes significant.



Relativistic (projection) effects

- The relativistic description of galaxy clustering can be derived from the fact that the number $dN_g^{\rm obs}$ of observed galaxies in a small volume is conserved:

$$dN_g^{\rm obs} = n_g^{\rm obs} dV_{\rm obs} = n_g^{\rm phy} dV_{\rm phy}$$

1) $dV_{\rm phy} \neq dV_{\rm obs}$ due to the distortion between these two volume elements: **the volume effect**, i.e.

- the redshift-space distortions **- the gravitational lensing**

2) Evolution bias b_e takes into account that the comoving number density of galaxies in the sample changes with redshift.

3) Magnification bias $\mathcal{Q}$: lensing magnification alters the observed number density of galaxies.

Relativistic (projection) effects

Putting all these effects together, the observed galaxy fluctuation is

$$\begin{aligned}\Delta_g^{(1)} &= \delta_g^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel^2 v + \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\partial_\parallel v - \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 4\mathcal{Q} + 1 - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\Phi \\ &\quad -2\left(1-\mathcal{Q}\right)\kappa^{(1)} + \frac{1}{\mathcal{H}}\Phi' + 2\left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]I^{(1)} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}}T^{(1)} \\ &= \sum_{a=1}^{8}\Delta_g^{a(1)}\end{aligned}$$

See Yoo 2008, Yoo et al 2009, Challinor et al. 2011, Bonvin & Durrer 2011 and Jeong et al. 2011

Relativistic (projection) effects

Putting all these effects together, the observed galaxy fluctuation is

$$
\begin{aligned}
\Delta_g^{(1)} &= \delta_g^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel^2 v + \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\partial_\parallel v - \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 4\mathcal{Q} + 1 - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\Phi \\
&\quad -2\left(1-\mathcal{Q}\right)\kappa^{(1)} + \frac{1}{\mathcal{H}}\Phi' + 2\left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right] I^{(1)} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}} T^{(1)} \\
&= \sum_{a=1}^{8} \Delta_g^{a(1)}
\end{aligned}
$$

Standard redshift-space distortions term

Doppler term

See Yoo 2008, Yoo et al 2009, Challinor et al. 2011, Bonvin & Durrer 2011 and Jeong et al. 2011

Relativistic (projection) effects

Putting all these effects together, the observed galaxy fluctuation is

$$\begin{aligned}\Delta_g^{(1)} &= \delta_g^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel^2 v + \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\partial_\parallel v - \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 4\mathcal{Q} + 1 - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\Phi \\ &\quad -2\,(1-\mathcal{Q})\,\kappa^{(1)} + \frac{1}{\mathcal{H}}\Phi' + 2\left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right] I^{(1)} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}} T^{(1)} \\ &= \sum_{a=1}^{8} \Delta_g^{a(1)}\end{aligned}$$

$I^{(1)} = -\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,\Phi'$ → is the integrated Sachs-Wolfe (ISW) effect at first order;

$T^{(1)} = -2\int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\Phi$ → is the (Shapiro) time-delay term

$\kappa^{(1)} = \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\,\tilde{\nabla}_\perp^2\Phi$ → is the weak-lensing convergence term;

See Yoo 2008, Yoo et al 2009, Challinor et al. 2011, Bonvin & Durrer 2011 and Jeong et al. 2011

At second order???

At second order:

see D.B.(2014)

$$
\begin{aligned}
\Delta_g^{(2)} &= \delta_g^{(2)} + \Big[b_e - 2\mathcal{Q} - \frac{\mathcal{H}'}{\mathcal{H}^2} - (1-\mathcal{Q})\,\frac{2}{\bar\chi\mathcal{H}}\Big]\,\Delta\ln a^{(2)} - (1-\mathcal{Q})\Big(2\Psi^{(2)} + \frac{1}{2}\hat h_\parallel^{(2)}\Big) - (1-\mathcal{Q})\,\frac{2}{\bar\chi}T^{(2)} - 2\,(1-\mathcal{Q})\,\kappa^{(2)} \\
&+\Phi^{(2)} + \frac{1}{\mathcal{H}}\Psi^{(2)\prime} - \frac{1}{2\mathcal{H}}\hat h_\parallel^{(2)\prime} - \frac{1}{\mathcal{H}}\partial_\parallel^2 v^{(2)} - \frac{1}{\mathcal{H}}\partial_\parallel \hat v_\parallel^{(2)} + 2\,(-1+2\mathcal{Q})\,\Phi\delta_g^{(1)} - \frac{2}{\mathcal{H}}\delta_g^{(1)}\partial_\parallel^2 v + \frac{2}{\mathcal{H}}\delta_g^{(1)}\Phi' + \frac{2}{\mathcal{H}}\Big(2\mathcal{Q} + \frac{\mathcal{H}'}{\mathcal{H}^2}\Big)\,\Phi\Phi' \\
&+\Big(-5 + 4\mathcal{Q} + 4\mathcal{Q}^2 - 4\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\,\Phi^2 + \left(\partial_\parallel v\right)^2 - \frac{2}{\mathcal{H}}\Big(1 + 2\mathcal{Q} + \frac{\mathcal{H}'}{\mathcal{H}^2}\Big)\,\Phi\partial_\parallel^2 v + \frac{2}{\mathcal{H}^2}\left(\Phi'\right)^2 + \frac{2}{\mathcal{H}^2}\left(\partial_\parallel^2 v\right)^2 - \frac{2}{\mathcal{H}}\Phi\partial_\parallel\Phi \\
&+\frac{2}{\mathcal{H}^2}\partial_\parallel v\partial_\parallel^2\Phi + \frac{4}{\mathcal{H}}\partial_\parallel v\partial_\parallel\Phi - \frac{2}{\mathcal{H}^2}\Phi\partial_\parallel^3 v + \frac{2}{\mathcal{H}^2}\Phi\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi} - \frac{2}{\mathcal{H}^2}\partial_\parallel v\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi} + \frac{2}{\mathcal{H}}\Big(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\Big)\,\partial_\parallel v\partial_\parallel^2 v - \frac{2}{\mathcal{H}^2}\Phi\partial_\parallel^2\Phi \\
&+\frac{2}{\mathcal{H}}\Big(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\Big)\,\partial_\parallel v\Phi' - \frac{4}{\mathcal{H}^2}\partial_\parallel^2 v\Phi' + \frac{2}{\mathcal{H}^2}\partial_\parallel v\partial_\parallel^3 v + \frac{2}{\mathcal{H}}\partial_{\perp i}v\partial_\perp^i\Phi - \frac{4}{\mathcal{H}}\partial_{\perp i}v\partial_\perp^i\partial_\parallel v + \Big(-1 + \frac{4}{\bar\chi\mathcal{H}}\Big)\,\partial_{\perp i}v\partial_\perp^i v \\
&+\Big\{\Big[-2b_e - 4\mathcal{Q} + 4b_e\mathcal{Q} - 8\mathcal{Q}^2 + 8\frac{\partial\mathcal{Q}}{\partial\ln\bar L} + 4\frac{\partial\mathcal{Q}}{\partial\ln\bar a} + 2\frac{\mathcal{H}'}{\mathcal{H}^2}(1-2\mathcal{Q}) + \frac{4}{\bar\chi\mathcal{H}}\Big(-1 + \mathcal{Q} + 2\mathcal{Q}^2 - 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\Big]\,\Phi \\
&+2\Big[b_e - 2\mathcal{Q} - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\Big]\,\delta_g^{(1)} - \frac{2}{\mathcal{H}}\frac{\mathrm{d}\delta_g^{(1)}}{\mathrm{d}\bar\chi} + \frac{2}{\mathcal{H}}\Big[-b_e + 2\mathcal{Q} + \frac{\mathcal{H}'}{\mathcal{H}^2} + \frac{2}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\Big]\,\partial_\parallel^2 v \\
&+\frac{2}{\mathcal{H}}\Big[-2 + b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{2}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\Big]\,\Phi' - \frac{4}{\mathcal{H}}\mathcal{Q}\partial_\parallel\Phi + 4\Big[-\Big(b_e - b_e\mathcal{Q} + 2\mathcal{Q}^2 - 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L} - \frac{\partial\mathcal{Q}}{\partial\ln\bar a}\Big) + \frac{\mathcal{H}'}{\mathcal{H}^2}(1-\mathcal{Q}) \\
&+\frac{1}{\bar\chi\mathcal{H}}\Big(1 - \mathcal{Q} + 2\mathcal{Q}^2 - 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\Big]\,\Big(\frac{1}{\bar\chi}T^{(1)} + \kappa^{(1)}\Big)\Big\}\,\Delta\ln a^{(1)} + \Big\{-b_e + b_e^2 + \frac{\partial b_e}{\partial\ln\bar a} + 6\mathcal{Q} - 4\mathcal{Q}b_e + 4\mathcal{Q}^2 \\
&-4\frac{\partial\mathcal{Q}}{\partial\ln\bar L} - 4\frac{\partial\mathcal{Q}}{\partial\ln\bar a} + (1 - 2b_e + 4\mathcal{Q})\,\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + 3\Big(\frac{\mathcal{H}'}{\mathcal{H}^2}\Big)^2 + \frac{6}{\bar\chi}\frac{\mathcal{H}'}{\mathcal{H}^3}(1-\mathcal{Q}) + \frac{2}{\bar\chi^2\mathcal{H}^2}\Big(1 - \mathcal{Q} + 2\mathcal{Q}^2 - 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big) \\
&+\frac{2}{\bar\chi\mathcal{H}}\Big[1 - 2b_e - \mathcal{Q} + 2b_e\mathcal{Q} - 4\mathcal{Q}^2 + 4\frac{\partial\mathcal{Q}}{\partial\ln\bar L} + 2\frac{\partial\mathcal{Q}}{\partial\ln\bar a}\Big]\Big\}\,\left(\Delta\ln a^{(1)}\right)^2 + 4\Big[+\frac{1}{\mathcal{H}}\Big(1 - \frac{\mathcal{H}'}{\mathcal{H}^2}\Big)\,\Phi' + \frac{1}{\mathcal{H}}\partial_\parallel\Phi \\
&+\frac{1}{\mathcal{H}}\Big(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\Big)\,\partial_\parallel^2 v + \frac{1}{\mathcal{H}^2}\partial_\parallel^2\Phi + \frac{1}{\mathcal{H}^2}\partial_\parallel^3 v - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi}\Big]\,I^{(1)} + \Big[-\frac{4}{\bar\chi}(1-\mathcal{Q})\,\delta_g^{(1)} - 2\partial_\parallel\delta_g^{(1)} - \frac{4}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\,\Phi' \\
&+\frac{4}{\bar\chi}\Big(-1 + \mathcal{Q} + 2\mathcal{Q}^2 - 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\,\Phi + 2\,(1-2\mathcal{Q})\,\partial_\parallel\Phi + \frac{4}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\,\partial_\parallel^2 v + \frac{2}{\mathcal{H}}\partial_\parallel^3 v - \frac{2}{\mathcal{H}}\partial_\parallel\Phi'\Big]\,T^{(1)} + \Big(1 - \mathcal{Q} + 2\mathcal{Q}^2 \\
&-2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\Big[\frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2 + \frac{4}{\bar\chi}T^{(1)}\kappa^{(1)}\Big] + 4\Big[-\Big(1 - \mathcal{Q} - 2\mathcal{Q}^2 + 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\,\Phi + \frac{1}{\mathcal{H}}(1-\mathcal{Q})\,\partial_\parallel^2 v - \frac{1}{\mathcal{H}}(1-\mathcal{Q})\,\Phi' \\
&-(1-\mathcal{Q})\,\delta_g^{(1)}\Big]\,\kappa^{(1)} + (1-\mathcal{Q})\,\vartheta_{ij}^{(1)}\vartheta^{ij(1)} + 2\Big(1 - \mathcal{Q} + 2\mathcal{Q}^2 - 2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\left(\kappa^{(1)}\right)^2 - 2\,(1-\mathcal{Q})\,|\gamma^{(1)}|^2 + 4\Big[\frac{\bar\chi}{\mathcal{H}}\Big(\partial_{\perp i}\Phi' \\
&-\partial_{\perp i}\partial_\parallel^2 v\Big) + \bar\chi\partial_{\perp i}\delta_g^{(1)} + \bar\chi\partial_{\perp i}\Phi - 2\bar\chi\,(1-\mathcal{Q})\,\partial_{\perp i}\Phi + \frac{1}{\mathcal{H}}(1-\mathcal{Q})\,\partial_{\perp i}\Delta\ln a^{(1)}\Big]\,S_\perp^{i(1)} - 4(1-\mathcal{Q})S_\perp^{i(1)}S_\perp^{j(1)}\delta_{ij} \\
&+2\Big[\frac{2}{\bar\chi\mathcal{H}}\partial_{\perp i}v - \frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\Phi' + \frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\partial_\parallel^2 v - \frac{2}{\mathcal{H}}\partial_{\perp i}\partial_\parallel v \; - \bar\chi\partial_{\perp i}\delta_g^{(1)} - \bar\chi\partial_{\perp i}\Phi + 2\bar\chi\,(1-\mathcal{Q})\,\partial_{\perp i}\Phi\Big]\,\partial_\perp^i T^{(1)} \\
&+4\mathcal{Q}^{(1)}\Big[\Phi - \Big(1 - \frac{1}{\bar\chi\mathcal{H}}\Big)\,\Delta\ln a^{(1)} + \frac{1}{\bar\chi}T^{(1)} + \kappa^{(1)}\Big] + 8(1-\mathcal{Q})\Big\{\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Big[-\Phi\tilde\partial_{\perp m}S_\perp^{m(1)} + \Big(\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde\chi} - \frac{1}{\tilde\chi}\Phi\Big)\,\kappa^{(1)}\Big] \\
&-\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\Big(\Phi^2 + \Phi' T^{(1)} + 2\Phi\kappa^{(1)} + \tilde\chi\tilde\partial_{\perp i}\Phi\partial_\perp^i T^{(1)}\Big) + \frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi - \tilde\chi)\Big[-2\Phi\tilde\partial_{\perp m}S_\perp^{m(1)} + 2\Big(\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde\chi} - \frac{1}{\tilde\chi}\Phi\Big)\,\kappa^{(1)}\Big]\Big\} \\
&-(1-\mathcal{Q})\left(24\Phi_o v_{\parallel o} + v_{\perp k\, o}v_{\perp o}^k\right) - 8(1-\mathcal{Q})v_{\parallel o}\Big(\frac{1}{\bar\chi}T^{(1)} + 3\int_0^{\bar\chi}\frac{\mathrm{d}\tilde\chi}{\tilde\chi}\Phi\Big) + 2\left(\Phi_o - v_{\parallel o}\right)\Big[\Big(\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\Big)\,\partial_\parallel^2 v \\
&+\Big(-\frac{\mathcal{H}'}{\mathcal{H}^3} + \frac{1}{\mathcal{H}}\Big)\,\Phi' + \frac{1}{\mathcal{H}^2}\partial_\parallel^2\Phi + \frac{1}{\mathcal{H}^2}\partial_\parallel^3 v + \frac{1}{\mathcal{H}}\partial_\parallel\Phi - \frac{1}{\mathcal{H}^2}\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi}\Big] + v_{\perp o}^i\Big[-2\bar\chi\partial_{\perp i}\Phi + 2\frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\left(-\Phi' + \partial_\parallel^2 v\right) - 2\bar\chi\partial_{\perp i}\delta_g^{(1)}\Big] \\
&+(1-\mathcal{Q})\,v_{\perp i\, o}\Big[4S_\perp^i - \frac{2}{\mathcal{H}}\partial_\perp^i\Delta\ln a^{(1)} + 4\bar\chi\partial_\perp^i\Phi\Big]\,. \qquad (100)
\end{aligned}
$$

See also D.B. et al.(2014a,b), Yoo & Zaldarriaga (2014), Di Dio et al (2014,2015)

At second order:

see D.B.(2014)

$$\Delta_g^{(2)}=\delta_g^{(2)}+\left[b_e-2\mathcal{Q}-\frac{\mathcal{H}'}{\mathcal{H}^2}-(1-\mathcal{Q})\frac{2}{\bar\chi\mathcal{H}}\right]\Delta\ln a^{(2)}-(1-\mathcal{Q})\left(2\Psi^{(2)}+\frac{1}{2}\hat h_\parallel^{(2)}\right)-(1-\mathcal{Q})\frac{2}{\bar\chi}T^{(2)}-2(1-\mathcal{Q})\kappa^{(2)}$$

$$+\Phi^{(2)}+\frac{1}{\mathcal{H}}\Psi^{(2)\prime}-\frac{1}{2\mathcal{H}}\hat h_\parallel^{(2)\prime}-\frac{1}{\mathcal{H}}\partial_\parallel^2 v^{(2)}-\frac{1}{\mathcal{H}}\partial_\parallel\hat v_\parallel^{(2)}+2(-1+2\mathcal{Q})\Phi\delta_g^{(1)}-\frac{2}{\mathcal{H}}\delta_g^{(1)}\partial_\parallel^2 v+\frac{2}{\mathcal{H}}\delta_g^{(1)}\Phi'+\frac{2}{\mathcal{H}}\left(2\mathcal{Q}+\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Phi\Phi'$$

$$+\left(-5+4\mathcal{Q}+4\mathcal{Q}^2-4\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\right)\Phi^2+(\partial_\parallel v)^2-\frac{2}{\mathcal{H}}\left(1+2\mathcal{Q}+\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\Phi\partial_\parallel^2 v+\frac{2}{\mathcal{H}^2}(\Phi')^2+\frac{2}{\mathcal{H}^2}\left(\partial_\parallel^2 v\right)^2-\frac{2}{\mathcal{H}}\Phi\partial_\parallel\Phi$$

$$+\frac{2}{\mathcal{H}^2}\partial_\parallel v\partial_\parallel^2\Phi+\frac{4}{\mathcal{H}}\partial_\parallel v\partial_\parallel\Phi-\frac{2}{\mathcal{H}^2}\Phi\partial_\parallel^3 v+\frac{2}{\mathcal{H}^2}\Phi\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi}-\frac{2}{\mathcal{H}^2}\partial_\parallel v\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi}+\frac{2}{\mathcal{H}}\left(1+\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\partial_\parallel v\partial_\parallel^2 v-\frac{2}{\mathcal{H}^2}\Phi\partial_\parallel^2\Phi$$

$$+\frac{2}{\mathcal{H}}\left(1-\frac{\mathcal{H}'}{\mathcal{H}^2}\right)\partial_\parallel v\Phi'-\frac{4}{\mathcal{H}^2}\partial_\parallel^2 v\Phi'+\frac{2}{\mathcal{H}^2}\partial_\parallel v\partial_\parallel^3 v+\frac{2}{\mathcal{H}}\partial_{\perp i}v\partial_\perp^i\Phi-\frac{4}{\mathcal{H}}\partial_{\perp i}v\partial_\perp^i\partial_\parallel v+\left(-1+\frac{4}{\bar\chi\mathcal{H}}\right)\partial_{\perp i}v\partial_\perp^i v$$

$$+\Big\{\Big[-2b_e-4\mathcal{Q}+4b_e\mathcal{Q}-8\mathcal{Q}^2+8\frac{\partial\mathcal{Q}}{\partial\ln\bar L}+4\frac{\partial\mathcal{Q}}{\partial\ln\bar a}+2\frac{\mathcal{H}'}{\mathcal{H}^2}(1-2\mathcal{Q})+\frac{4}{\bar\chi\mathcal{H}}\left(-1+\mathcal{Q}+2\mathcal{Q}^2-2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\right)\Big]\Phi$$

$$+2\left[b_e-2\mathcal{Q}-\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{2}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\right]\delta_g^{(1)}-\frac{2}{\mathcal{H}}\frac{\mathrm{d}\delta_g^{(1)}}{\mathrm{d}\bar\chi}+\frac{2}{\mathcal{H}}\left[-b_e+2\mathcal{Q}+\frac{\mathcal{H}'}{\mathcal{H}^2}+\frac{2}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\right]\partial_\parallel^2 v$$

$$+\frac{2}{\mathcal{H}}\left[-2+b_e-\frac{\mathcal{H}'}{\mathcal{H}^2}-\frac{2}{\bar\chi\mathcal{H}}(1-\mathcal{Q})\right]\Phi'-\frac{4}{\mathcal{H}}\mathcal{Q}\partial_\parallel\Phi+4\Big[-\left(b_e-b_e\mathcal{Q}+2\mathcal{Q}^2-2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}-\frac{\partial\mathcal{Q}}{\partial\ln\bar a}\right)+\frac{\mathcal{H}'}{\mathcal{H}^2}(1-\mathcal{Q})$$

$$+\frac{1}{\bar\chi\mathcal{H}}\left(1-\mathcal{Q}+2\mathcal{Q}^2-2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\right)\Big]\Big]\left(\frac{1}{\bar\chi}T^{(1)}+\kappa^{(1)}\right)\Big\}\Delta\ln a^{(1)}+\Big\{-b_e+b_e^2+\frac{\partial b_e}{\partial\ln\bar a}+6\mathcal{Q}-4\mathcal{Q}b_e+4\mathcal{Q}^2$$

Including all redshift effects: lensing distortions from convergence and shear, and contributions from velocities, Sachs-Wolfe, integrated SW, magnification and time-delay terms.

$$-2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\Big)\left[\frac{2}{\bar\chi^2}\left(T^{(1)}\right)^2+\frac{4}{\bar\chi}T^{(1)}\kappa^{(1)}\right]+4\Big[-\left(1-\mathcal{Q}-2\mathcal{Q}^2+2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\right)\Phi+\frac{1}{\mathcal{H}}(1-\mathcal{Q})\partial_\parallel^2 v-\frac{1}{\mathcal{H}}(1-\mathcal{Q})\Phi'$$

$$-(1-\mathcal{Q})\delta_g^{(1)}\Big]\kappa^{(1)}+(1-\mathcal{Q})\vartheta_{ij}^{(1)}\vartheta^{ij(1)}+2\left(1-\mathcal{Q}+2\mathcal{Q}^2-2\frac{\partial\mathcal{Q}}{\partial\ln\bar L}\right)\left(\kappa^{(1)}\right)^2-2(1-\mathcal{Q})\left|\gamma^{(1)}\right|^2+4\Big[\frac{\bar\chi}{\mathcal{H}}\Big(\partial_{\perp i}\Phi'$$

$$-\partial_{\perp i}\partial_\parallel^2 v\Big)+\bar\chi\partial_{\perp i}\delta_g^{(1)}+\bar\chi\partial_{\perp i}\Phi-2\bar\chi(1-\mathcal{Q})\partial_{\perp i}\Phi+\frac{1}{\mathcal{H}}(1-\mathcal{Q})\partial_{\perp i}\Delta\ln a^{(1)}\Big]S_\perp^{i(1)}-4(1-\mathcal{Q})S_\perp^{i(1)}S_\perp^{j(1)}\delta_{ij}$$

$$+2\left[\frac{2}{\bar\chi\mathcal{H}}\partial_{\perp i}v-\frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\Phi'+\frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\partial_\parallel^2 v-\frac{2}{\mathcal{H}}\partial_{\perp i}\partial_\parallel v-\bar\chi\partial_{\perp i}\delta_g^{(1)}-\bar\chi\partial_{\perp i}\Phi+2\bar\chi(1-\mathcal{Q})\partial_{\perp i}\Phi\right]\partial_\perp^i T^{(1)}$$

$$+4\mathcal{Q}^{(1)}\left[\Phi-\left(1-\frac{1}{\bar\chi\mathcal{H}}\right)\Delta\ln a^{(1)}+\frac{1}{\bar\chi}T^{(1)}+\kappa^{(1)}\right]+8(1-\mathcal{Q})\Big\{\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left[-\Phi\tilde\partial_{\perp m}S_\perp^{m(1)}+\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde\chi}-\frac{1}{\tilde\chi}\Phi\right)\kappa^{(1)}\right]$$

$$-\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\left(\Phi^2+\Phi'T^{(1)}+2\Phi\kappa^{(1)}+\tilde\chi\tilde\partial_{\perp i}\Phi\partial_\perp^i T^{(1)}\right)+\frac{1}{\bar\chi}\int_0^{\bar\chi}\mathrm{d}\tilde\chi\,(\bar\chi-\tilde\chi)\left[-2\Phi\tilde\partial_{\perp m}S_\perp^{m(1)}+2\left(\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde\chi}-\frac{1}{\tilde\chi}\Phi\right)\kappa^{(1)}\right]\Big\}$$

$$-(1-\mathcal{Q})\left(24\Phi_o v_{\parallel o}+v_{\perp k\,o}v_{\perp o}^k\right)-8(1-\mathcal{Q})v_{\parallel o}\left(\frac{1}{\bar\chi}T^{(1)}+3\int_0^{\bar\chi}\frac{\mathrm{d}\tilde\chi}{\tilde\chi}\Phi\right)+2\left(\Phi_o-v_{\parallel o}\right)\Big[\left(\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\right)\partial_\parallel^2 v$$

$$+\left(-\frac{\mathcal{H}'}{\mathcal{H}^3}+\frac{1}{\mathcal{H}}\right)\Phi'+\frac{1}{\mathcal{H}^2}\partial_\parallel^2\Phi+\frac{1}{\mathcal{H}^2}\partial_\parallel^3 v+\frac{1}{\mathcal{H}}\partial_\parallel\Phi-\frac{1}{\mathcal{H}^2}\frac{\mathrm{d}\Phi'}{\mathrm{d}\bar\chi}\Big]+v_{\perp o}^i\left[-2\bar\chi\partial_{\perp i}\Phi+2\frac{\bar\chi}{\mathcal{H}}\partial_{\perp i}\left(-\Phi'+\partial_\parallel^2 v\right)-2\bar\chi\partial_{\perp i}\delta_g^{(1)}\right]$$

$$+(1-\mathcal{Q})v_{\perp i o}\left[4S_\perp^i-\frac{2}{\mathcal{H}}\partial_\perp^i\Delta\ln a^{(1)}+4\bar\chi\partial_\perp^i\Phi\right]. \qquad (100)$$

See also D.B. et al.(2014a,b), Yoo & Zaldarriaga (2014), Di Dio et al (2014,2015)

The Spherical Bessel representation of galaxy fractional overdensity

- Δ_g can be decomposed in the following way:

$$\Delta^g_{\ell m}(k) = \langle k\ell m|\Delta_g\rangle = \int \mathrm{d}^3\mathbf{x}\ \langle k\ell m|\mathbf{x}\rangle\langle\mathbf{x}|\Delta_g\rangle$$

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- Then at first and second order we find

$$\Delta^g_{\ell m}(k) = \Delta^{g(1)}_{\ell m}(k) + \frac{1}{2}\Delta^{g(2)}_{\ell m}(k) + \ldots = \sum_b \Delta^{b(1)}_{\ell m}(k) + \frac{1}{2}\sum_a \Delta^{a(2)}_{\ell m}(k) + \ldots$$

see also Yoo & Desjacques(2013)

First-order terms

$$\Delta^{b(1)}_{\ell m}(k) = \langle k\ell m|\Delta^{b(1)}\rangle = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle \Delta^{b(1)}_{g}(\mathbf{x})$$

see also Yoo & Desjacques (2013)

First-order terms

$$\Delta_{\ell m}^{b(1)}(k) = \langle k\ell m | \Delta^{b(1)} \rangle = \int \mathrm{d}^3\mathbf{x}\, \mathcal{W}(\bar{\chi})\ \langle k\ell m | \mathbf{x} \rangle \Delta_g^{b(1)}(\mathbf{x})$$

where:

$$\Delta_g^{b(1)}(\mathbf{x}) = \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\ \mathbb{W}^b \left(\bar{\chi}, \tilde{\chi}, \eta, \tilde{\eta}, \frac{\partial}{\partial \tilde{\chi}}, \frac{\partial}{\partial \tilde{\eta}}, \triangle_{\hat{\mathbf{n}}} \right) \left[\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathcal{T}^b(\mathbf{k}, \tilde{\eta}) \Phi_{\mathrm{p}}(\mathbf{k}) e^{i\mathbf{k}\tilde{\mathbf{x}}} \right]$$

- $\mathcal{T}^b(\mathbf{k}, \eta)$ is a generalised transfer function which relates the linear primordial potential with a generic perturbation term (labeled with b);
- $\Phi_{\mathrm{p}}(\mathbf{k})$ is the primordial potential set during the inflation epoch:
- $\mathbb{W}^b$ is a generic operator that depends on χ, η, $\partial/\partial\chi$, $\partial/\partial\eta$ and $\Delta_{\mathbf{n}}$

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First-order terms

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where

$$\mathbb{W}^b\left(\bar{\chi}, \tilde{\chi}, \eta, \tilde{\eta}, \frac{\partial}{\partial\tilde{\chi}}, \frac{\partial}{\partial\tilde{\eta}}, \triangle_{\hat{\mathbf{n}}}\right) Y_{\ell m}(\hat{\mathbf{n}}) = \mathbb{W}^b_{\ell}\left(\bar{\chi}, \tilde{\chi}, \eta, \tilde{\eta}, \frac{\partial}{\partial\tilde{\chi}}, \frac{\partial}{\partial\tilde{\eta}}\right) Y_{\ell m}(\hat{\mathbf{n}})$$

$$\triangle_{\hat{\mathbf{n}}} Y_{\ell m}(\hat{\mathbf{n}}) = -\ell(\ell+1) Y_{\ell m}(\hat{\mathbf{n}})$$

see also Yoo & Desjacques(2013)

First-order terms

$$\Delta^{b(1)}_{\ell m}(k) = \langle k\ell m|\Delta^{b(1)}\rangle = \int \mathrm{d}^3\mathbf{x}\, \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle \Delta^{b(1)}_g(\mathbf{x})$$

where:

$$\Delta^{b(1)}_g(\mathbf{x}) = \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\ \mathbb{W}^b\left(\bar{\chi}, \tilde{\chi}, \eta, \tilde{\eta}, \frac{\partial}{\partial\tilde{\chi}}, \frac{\partial}{\partial\tilde{\eta}}, \triangle_{\hat{\mathbf{n}}}\right)\left[\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\mathcal{T}^b(\mathbf{k},\tilde{\eta})\Phi_{\mathrm{p}}(\mathbf{k})e^{i\mathbf{k}\tilde{\mathbf{x}}}\right]$$

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$$\mathbb{W}^b\left(\bar{\chi}, \tilde{\chi}, \eta, \tilde{\eta}, \frac{\partial}{\partial\tilde{\chi}}, \frac{\partial}{\partial\tilde{\eta}}, \triangle_{\hat{\mathbf{n}}}\right)\left[\mathcal{T}^b(\mathbf{k},\tilde{\eta})e^{i\mathbf{k}\tilde{\mathbf{x}}}\right] = \sum_{\ell m} 4\pi i^\ell \left[\mathbb{W}^b_\ell\left(\bar{\chi}, \tilde{\chi}, \eta, \tilde{\eta}, \frac{\partial}{\partial\tilde{\chi}}, \frac{\partial}{\partial\tilde{\eta}}\right)\mathcal{T}^b(\mathbf{k},\tilde{\eta})j_\ell(k\tilde{\chi})\right] Y^*_{\ell m}(\hat{\mathbf{k}})Y_{\ell m}(\hat{\mathbf{n}})$$

see also Yoo & Desjacques (2013)

First-order terms

$$\Delta_{\ell m}^{b(1)}(k) = \langle k\ell m|\Delta^{b(1)}\rangle = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle \Delta_g^{b(1)}(\mathbf{x})$$

$$\Downarrow$$

$$\Delta_{\ell m}^{b(1)}(k) = \int \frac{\mathrm{d}^3\tilde{\mathbf{k}}}{(2\pi)^3} \mathcal{M}_\ell^{b(1)}(k,\tilde{k}) Y_{\ell m}^*(\hat{\tilde{\mathbf{k}}}) \Phi_{\mathrm{p}}(\tilde{\mathbf{k}})$$

where:

$$\mathcal{M}_\ell^{b(1)}(k,\tilde{k}) = k\sqrt{\frac{2}{\pi}} \int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell) j_\ell(k\bar{\chi}) \int_0^{\bar{\chi}} \mathrm{d}\tilde{\chi}\ \left[\mathbb{W}_\ell^b\left(\bar{\chi},\tilde{\chi},\eta,\tilde{\eta},\frac{\partial}{\partial\tilde{\chi}},\frac{\partial}{\partial\tilde{\eta}}\right)\mathcal{T}^b(\tilde{\mathbf{k}},\tilde{\eta}) j_\ell(\tilde{k}\tilde{\chi})\right]$$

see also Yoo & Desjacques (2013)

First-order terms

$$\Delta^{b(1)}_{\ell m}(k) = \langle k\ell m|\Delta^{b(1)}\rangle = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle \Delta^{b(1)}_{g}(\mathbf{x})$$

$$\Downarrow$$

$$\Delta^{b(1)}_{\ell m}(k) = \int \frac{\mathrm{d}^3\tilde{\mathbf{k}}}{(2\pi)^3} \mathcal{M}^{b(1)}_{\ell}(k,\tilde{k}) Y^{*}_{\ell m}(\hat{\tilde{\mathbf{k}}}) \Phi_{\mathrm{p}}(\tilde{\mathbf{k}})$$

$$\Downarrow$$

$$\langle \Delta^{b(1)*}_{\ell m}(k)\ \Delta^{c(1)}_{\ell' m'}(k')\rangle = \delta_{\ell\ell'}\delta_{mm'} \int \frac{\tilde{k}^2\mathrm{d}\tilde{k}}{(2\pi)^3}\ \mathcal{M}^{b(1)*}_{\ell}(k,\tilde{k})\ \mathcal{M}^{c(1)}_{\ell}(k',\tilde{k})\ P_{\Phi}(\tilde{k})$$

Spherical galaxy Power Spectrum !

First-order terms

$$\mathcal{M}_\ell^{b(1)}(k,\tilde{k}) = k\sqrt{\frac{2}{\pi}}\int \mathrm{d}\bar{\chi}\;\bar{\chi}^2\mathcal{W}(\bar{\chi})(4\pi i^\ell)j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\;\left[\mathbb{W}_\ell^b\left(\bar{\chi},\tilde{\chi},\eta,\tilde{\eta},\frac{\partial}{\partial\tilde{\chi}},\frac{\partial}{\partial\tilde{\eta}}\right)\mathcal{T}^b(\tilde{\mathbf{k}},\tilde{\eta})j_\ell(\tilde{k}\tilde{\chi})\right]$$

- Example 1):

$$\delta_g^{(1)}(\mathbf{k},\eta) = \mathcal{T}^{\delta_g}(\mathbf{k},\eta)\Phi_{\mathrm{p}}(\mathbf{k})$$

$$\Downarrow$$

$$\mathbb{W}^b \;\Rightarrow\; \delta^D(\bar{\chi}-\tilde{\chi})$$

$$\Downarrow$$

$$\mathcal{M}_\ell^{\delta_g(1)}(k,\tilde{k}) = k\sqrt{\frac{2}{\pi}}\int \mathrm{d}\bar{\chi}\;\bar{\chi}^2\mathcal{W}(\bar{\chi})(4\pi i^\ell)j_\ell(k\bar{\chi})j_\ell(\tilde{k}\bar{\chi})\mathcal{T}^{\delta_g}(\tilde{\mathbf{k}},\eta)$$

First-order terms

$$\mathcal{M}_\ell^{b(1)}(k,\tilde{k}) = k\sqrt{\frac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2\mathcal{W}(\bar{\chi})(4\pi i^\ell)j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\ \left[\mathbb{W}_\ell^b\left(\bar{\chi},\tilde{\chi},\eta,\tilde{\eta},\frac{\partial}{\partial\tilde{\chi}},\frac{\partial}{\partial\tilde{\eta}}\right)\mathcal{T}^b(\tilde{\mathbf{k}},\tilde{\eta})j_\ell(\tilde{k}\tilde{\chi})\right]$$

- Example 2):

$$-\frac{1}{\mathcal{H}}\partial_\parallel^2 v \qquad \& \qquad v^{(1)}(\mathbf{k},\eta) = \mathcal{T}^v(\mathbf{k},\eta)\Phi_\mathrm{p}(\mathbf{k})$$

$$\Downarrow$$

$$\mathbb{W}^b \quad \Rightarrow \quad \delta(\bar{\chi}-\tilde{\chi})\left(-\frac{1}{\mathcal{H}}\right)\frac{\partial^2}{\partial\tilde{\chi}^2}$$

$$\Downarrow$$

$$\mathcal{M}_\ell^{\partial_\parallel^2 v(1)}(k,\tilde{k}) = k\sqrt{\frac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2\mathcal{W}(\bar{\chi})(4\pi i^\ell)\left[-\frac{1}{\mathcal{H}(\eta)}\right]j_\ell(k\bar{\chi})\left[\frac{\partial^2}{\partial\bar{\chi}^2}j_\ell(\tilde{k}\bar{\chi})\right]\mathcal{T}^v(\tilde{\mathbf{k}},\eta)$$

First-order terms

$$\mathcal{M}_\ell^{b(1)}(k,\tilde{k}) = k\sqrt{\frac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2\mathcal{W}(\bar{\chi})(4\pi i^\ell)j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\ \left[\mathbb{W}_\ell^b\left(\bar{\chi},\tilde{\chi},\eta,\tilde{\eta},\frac{\partial}{\partial\tilde{\chi}},\frac{\partial}{\partial\tilde{\eta}}\right)\mathcal{T}^b(\tilde{\mathbf{k}},\tilde{\eta})j_\ell(\tilde{k}\tilde{\chi})\right]$$

- Example 3):

$$-2\left(1-\mathcal{Q}\right)\kappa^{(1)}, \quad \kappa^{(1)} = \int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\,(\bar{\chi}-\tilde{\chi})\,\frac{\tilde{\chi}}{\bar{\chi}}\,\tilde{\nabla}_\perp^2\Phi\ , \quad \Phi^{(1)}(\mathbf{k},\eta) = \mathcal{T}^\Phi(\mathbf{k},\eta)\Phi_\mathrm{p}(\mathbf{k})$$

$$\Downarrow$$

$$\mathbb{W}^b \Rightarrow\ -2\left(1-\mathcal{Q}\right)\frac{(\bar{\chi}-\tilde{\chi})}{\bar{\chi}\tilde{\chi}}\triangle_{\hat{\mathbf{n}}} \qquad \triangle_{\hat{\mathbf{n}}}Y_{\ell m}(\hat{\mathbf{n}}) = -\ell(\ell+1)Y_{\ell m}(\hat{\mathbf{n}})$$

$$\Downarrow$$

$$\mathcal{M}_\ell^{\kappa(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2\mathcal{W}(\bar{\chi})[8\pi\ell(\ell+1)i^\ell][1-\mathcal{Q}(\eta)]j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\ \left[\tfrac{(\bar{\chi}-\tilde{\chi})}{\bar{\chi}\tilde{\chi}}j_\ell(\tilde{k}\tilde{\chi})\mathcal{T}^\Phi(\tilde{\mathbf{k}},\tilde{\eta})\right]$$

First-order terms

$$\Delta_g^{(1)} = \delta_g^{(1)} - \frac{1}{\mathcal{H}}\partial_\parallel^2 v + \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\partial_\parallel v - \left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 4\mathcal{Q} + 1 - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right]\Phi$$

$$-2\left(1-\mathcal{Q}\right)\kappa^{(1)} + \frac{1}{\mathcal{H}}\Phi' + 2\left[b_e - \frac{\mathcal{H}'}{\mathcal{H}^2} - 2\mathcal{Q} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}\mathcal{H}}\right] I^{(1)} - 2\frac{(1-\mathcal{Q})}{\bar{\chi}} T^{(1)}$$

$$\mathcal{M}_\ell^{\delta_g(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell) j_\ell(k\bar{\chi}) j_\ell(\tilde{k}\bar{\chi}) \mathcal{T}^{\delta_g}(\tilde{\mathbf{k}},\eta)\ ,$$

$$\mathcal{M}_\ell^{\partial_\parallel^2 v(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell)\left[-\tfrac{1}{\mathcal{H}(\eta)}\right] j_\ell(k\bar{\chi})\left[\tfrac{\partial^2}{\partial\bar{\chi}^2} j_\ell(\tilde{k}\bar{\chi})\right]\mathcal{T}^v(\tilde{\mathbf{k}},\eta)\ ,$$

$$\mathcal{M}_\ell^{\partial_\parallel v(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell)\left[b_e(\eta) - \tfrac{\mathcal{H}'(\eta)}{\mathcal{H}^2(\eta)} - 2\mathcal{Q}(\eta) - 2\tfrac{(1-\mathcal{Q}(\eta))}{\bar{\chi}\mathcal{H}(\eta)}\right] j_\ell(k\bar{\chi})\left[\tfrac{\partial}{\partial\bar{\chi}} j_\ell(\tilde{k}\bar{\chi})\right]\mathcal{T}^v(\tilde{\mathbf{k}},\eta)\ ,$$

$$\mathcal{M}_\ell^{\Phi(1)}(k,\tilde{k}) = -k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell)\left[b_e(\eta) - \tfrac{\mathcal{H}'(\eta)}{\mathcal{H}^2(\eta)} - 4\mathcal{Q}(\eta) + 1 - 2\tfrac{(1-\mathcal{Q}(\eta))}{\bar{\chi}\mathcal{H}(\eta)}\right] j_\ell(k\bar{\chi}) j_\ell(\tilde{k}\bar{\chi})\mathcal{T}^\Phi(\tilde{\mathbf{k}},\eta)\ ,$$

$$\mathcal{M}_\ell^{\Phi'(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell)\left[\tfrac{1}{\mathcal{H}(\eta)}\right] j_\ell(k\bar{\chi}) j_\ell(\tilde{k}\bar{\chi})\tfrac{\partial}{\partial\eta}\mathcal{T}^\Phi(\tilde{\mathbf{k}},\eta)\ ,$$

$$\mathcal{M}_\ell^{\kappa(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})[8\pi\ell(\ell+1)i^\ell][1-\mathcal{Q}(\eta)] j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\ \left[\tfrac{(\bar{\chi}-\tilde{\chi})}{\bar{\chi}\tilde{\chi}} j_\ell(\tilde{k}\tilde{\chi})\mathcal{T}^\Phi(\tilde{\mathbf{k}},\tilde{\eta})\right]\ ,$$

$$\mathcal{M}_\ell^{I(1)}(k,\tilde{k}) = -k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(4\pi i^\ell)\left[b_e(\eta) - \tfrac{\mathcal{H}'(\eta)}{\mathcal{H}^2(\eta)} - 2\mathcal{Q}(\eta) - 2\tfrac{(1-\mathcal{Q}(\eta))}{\bar{\chi}\mathcal{H}(\eta)}\right] j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\ \left[j_\ell(\tilde{k}\tilde{\chi})\tfrac{\partial}{\partial\tilde{\eta}}\mathcal{T}^\Phi(\tilde{\mathbf{k}},\tilde{\eta})\right]$$

$$\mathcal{M}_\ell^{T(1)}(k,\tilde{k}) = k\sqrt{\tfrac{2}{\pi}}\int \mathrm{d}\bar{\chi}\ \bar{\chi}^2 \mathcal{W}(\bar{\chi})(16\pi i^\ell)\left[\tfrac{1-\mathcal{Q}(\eta)}{\bar{\chi}}\right] j_\ell(k\bar{\chi})\int_0^{\bar{\chi}}\mathrm{d}\tilde{\chi}\ \left[j_\ell(\tilde{k}\tilde{\chi})\mathcal{T}^\Phi(\tilde{\mathbf{k}},\tilde{\eta})\right]\ .$$

Bispectrum including GR projection effects on large scales

- Schmidt et al. 2008 studied the corrections to the galaxy three-point correlation function induced by weak lensing magnification.
- Di Dio et al. (2014, 2015, 2016a, 2016b) and Kehagias et al. 2015 compute the three-point correlation function in configuration space, including only some projection terms.
- Umeh et al. (2016), Jolicoeur et al. (2017a,b, 2018) analyzed the bispectrum in Fourier space including some bias terms in flat-sky but not non-local integrated terms.
- Kazuya et al. +DB (2018), using the consistency relation in Fourier space, we derive the observed galaxy bispectrum from single-field inflation in the squeezed limit.

Bispectrum including GR projection effects on large scales

- In DB+ 2017 we write the galaxy Bispectrum using the “Spherical-Bessel” formalism
- Spherical galaxy Bispectrum includes all wide-angle and relativistic terms
- **The full expression is without any approximations**!!!

Second order

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle\ \frac{1}{2}\Delta^{a(2)}_{g}(\mathbf{x})$$

Second order

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle\ \frac{1}{2}\Delta^{a(2)}_{g}(\mathbf{x})$$

$$\Downarrow$$

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \sum_{\ell_{\mathbf{p}} m_{\mathbf{p}} \ell_{\mathbf{q}} m_{\mathbf{q}}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\widetilde{\mathcal{M}}^{a(2)}_{\ell m \ell_{\mathbf{p}} m_{\mathbf{p}} \ell_{\mathbf{q}} m_{\mathbf{q}}}(k;\mathbf{p},\mathbf{q}) Y^*_{\ell_{\mathbf{p}} m_{\mathbf{p}}}(\hat{\mathbf{p}}) Y^*_{\ell_{\mathbf{q}} m_{\mathbf{q}}}(\hat{\mathbf{q}}) \Phi_{\mathrm{p}}(\mathbf{p})\Phi_{\mathrm{p}}(\mathbf{q})$$

Second order

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle\ \frac{1}{2}\Delta^{a(2)}_{g}(\mathbf{x})$$

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \sum_{\ell_{\mathbf{p}} m_{\mathbf{p}} \ell_{\mathbf{q}} m_{\mathbf{q}}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\widetilde{\mathcal{M}}^{a(2)}_{\ell m \ell_{\mathbf{p}} m_{\mathbf{p}} \ell_{\mathbf{q}} m_{\mathbf{q}}}(k;\mathbf{p},\mathbf{q}) Y^{*}_{\ell_{\mathbf{p}} m_{\mathbf{p}}}(\hat{\mathbf{p}}) Y^{*}_{\ell_{\mathbf{q}} m_{\mathbf{q}}}(\hat{\mathbf{q}})\Phi_{\mathrm{p}}(\mathbf{p})\Phi_{\mathrm{p}}(\mathbf{q})$$

$$\widetilde{\mathcal{M}}^{a(2)}_{\ell m \ell_{\mathbf{p}} m_{\mathbf{p}} \ell_{\mathbf{q}} m_{\mathbf{q}}}(k;\mathbf{p},\mathbf{q}) = \sum_{\bar{\ell}\bar{m}} \mathcal{M}^{a(2)}_{\ell m \ell_{\mathbf{p}} m_{\mathbf{p}} \ell_{\mathbf{q}} m_{\mathbf{q}} \bar{\ell}\bar{m}}(k;p,q) Y_{\bar{\ell}\bar{m}}(\hat{\mathbf{p}}) Y^{*}_{\bar{\ell}\bar{m}}(\hat{\mathbf{q}})$$

where we have used

$$\mathcal{P}_{\bar{\ell}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}) = \frac{4\pi}{2\bar{\ell}+1}\sum_{\bar{m}=-\bar{\ell}}^{\bar{\ell}} Y_{\bar{\ell}\bar{m}}(\hat{\mathbf{p}}) Y^{*}_{\bar{\ell}\bar{m}}(\hat{\mathbf{q}})$$

Second order

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \int \mathrm{d}^3\mathbf{x}\ \mathcal{W}(\bar{\chi})\ \langle k\ell m|\mathbf{x}\rangle\ \frac{1}{2}\Delta^{a(2)}_{g}(\mathbf{x})$$

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \sum_{\ell_\mathbf{p} m_\mathbf{p} \ell_\mathbf{q} m_\mathbf{q}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \widetilde{\mathcal{M}}^{a(2)}_{\ell m \ell_\mathbf{p} m_\mathbf{p} \ell_\mathbf{q} m_\mathbf{q}}(k;\mathbf{p},\mathbf{q}) Y^*_{\ell_\mathbf{p} m_\mathbf{p}}(\hat{\mathbf{p}}) Y^*_{\ell_\mathbf{q} m_\mathbf{q}}(\hat{\mathbf{q}}) \Phi_\mathrm{p}(\mathbf{p})\Phi_\mathrm{p}(\mathbf{q})$$

$$\widetilde{\mathcal{M}}^{a(2)}_{\ell m \ell_\mathbf{p} m_\mathbf{p} \ell_\mathbf{q} m_\mathbf{q}}(k;\mathbf{p},\mathbf{q}) = \sum_{\bar{\ell}\bar{m}} \mathcal{M}^{a(2)}_{\ell m \ell_\mathbf{p} m_\mathbf{p} \ell_\mathbf{q} m_\mathbf{q} \bar{\ell}\bar{m}}(k;p,q) Y_{\bar{\ell}\bar{m}}(\hat{\mathbf{p}}) Y^*_{\bar{\ell}\bar{m}}(\hat{\mathbf{q}})$$

$$\frac{1}{2}\Delta^{a(2)}_{\ell m}(k) = \sum_{\ell_\mathbf{p} m_\mathbf{p} \ell_\mathbf{q} m_\mathbf{q} \bar{\ell}\bar{m}} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \mathcal{M}^{a(2)}_{\ell m \ell_\mathbf{p} m_\mathbf{p} \ell_\mathbf{q} m_\mathbf{q} \bar{\ell}\bar{m}}(k;p,q) Y^*_{\ell_\mathbf{p} m_\mathbf{p}}(\hat{\mathbf{p}}) Y_{\bar{\ell}\bar{m}}(\hat{\mathbf{p}}) Y^*_{\ell_\mathbf{q} m_\mathbf{q}}(\hat{\mathbf{q}}) Y^*_{\bar{\ell}\bar{m}}(\hat{\mathbf{q}}) \Phi_\mathrm{p}(\mathbf{p})\Phi_\mathrm{p}(\mathbf{q})$$

$\mathcal{M}^{(2)}$ are generating functions at second contain all the physical effects!

Bispectrum

$$\langle \Delta^{g}_{\ell_1 m_1}(k_1)\Delta^{g}_{\ell_2 m_2}(k_2)\Delta^{g}_{\ell_3 m_3}(k_2)\rangle = \left\langle \frac{1}{2}\Delta^{g(2)}_{\ell_1 m_1}(k_1)\Delta^{g(1)}_{\ell_2 m_2}(k_2)\Delta^{g(1)}_{\ell_3 m_3}(k_3)\right\rangle + \left\langle \Delta^{g(1)}_{\ell_1 m_1}(k_1)\frac{1}{2}\Delta^{g(2)}_{\ell_2 m_2}(k_2)\Delta^{g(1)}_{\ell_3 m_3}(k_3)\right\rangle$$
$$+ \left\langle \Delta^{g(1)}_{\ell_1 m_1}(k_1)\Delta^{g(1)}_{\ell_2 m_2}(k_2)\frac{1}{2}\Delta^{g(2)}_{\ell_3 m_3}(k_3)\right\rangle$$

Bispectrum

$$\langle \Delta^g_{\ell_1 m_1}(k_1)\Delta^g_{\ell_2 m_2}(k_2)\Delta^g_{\ell_3 m_3}(k_2)\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)$$

$B_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)$ is rotationally invariant for rotations (around the line of sight passing through the circumcenter), i.e. it must satisfy the following conditions:

(i) $\ell_j + \ell_k \geq \ell_i \geq |\ell_j - \ell_k|$ (triangle rule);

(ii) $\ell_1 + \ell_2 + \ell_3 =$ even;

(iii) $m_1 + m_2 + m_3 = 0$.

Bispectrum

$$\langle \Delta^g_{\ell_1 m_1}(k_1)\Delta^g_{\ell_2 m_2}(k_2)\Delta^g_{\ell_3 m_3}(k_2)\rangle = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3)$$

where

$$B_{\ell_1\ell_2\ell_3}(k_1,k_2,k_3) = \sum_{abc} \mathcal{C}^j_\ell \int \frac{q_2^2 \mathrm{d}q_2}{(2\pi)^3}\frac{q_3^2 \mathrm{d}q_3}{(2\pi)^3}\left[\mathcal{K}^{a(2)}_{\ell_1 \ell_{\mathbf{p}_1}\ell_{\mathbf{q}_1}\bar{\ell}_1}(k_1;q_2,q_3)\mathcal{M}^{b(1)}_{\ell_2}(k_2,q_2)\mathcal{M}^{c(1)}_{\ell_3}(k_3,q_3)\right] P_\Phi(q_2)P_\Phi(q_3) + \text{cyc}$$

with

$$\mathcal{M}^{a(2)}_{\ell_1 m_1 \ell_{\mathbf{p}_1} m_{\mathbf{p}_1} \ell_{\mathbf{q}_1} m_{\mathbf{q}_1} \bar{\ell}_1 \bar{m}_1}(k_1;q_2,q_3) = (-1)^{m_1}\mathcal{G}^{\ell_1 \ell_{\mathbf{p}_1}\ell_{\mathbf{q}_1}}_{-m_1 m_{\mathbf{p}_1} m_{\mathbf{q}_1}}\mathcal{K}^{a(2)}_{\ell_1 \ell_{\mathbf{p}_1}\ell_{\mathbf{q}_1}\bar{\ell}_1}(k_1;q_2,q_3) \quad ,$$

$\mathcal{C}^j_\ell$ is a combination of multipoles and 3j- and 6j-Wigner symbols, P_Φ denotes the primordial and the Gaunt integral is

$$\mathcal{G}^{\ell_1\ell_2\ell_3}_{m_1 m_2 m_3} = \int \mathrm{d}^2\hat{\mathbf{n}}\; Y_{\ell_1 m_1}(\hat{\mathbf{n}})Y_{\ell_2 m_2}(\hat{\mathbf{n}})Y_{\ell_3 m_3}(\hat{\mathbf{n}}) = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

Conclusions

- While it is always possible to embed the observed sphere in a cubic volume with rectangular coordinates and to perform a power spectrum analysis, it becomes difficult in principle to connect these large-scale measurements to the underlying theory, because the flat-sky approximation has a limited range of validity.
- The observed galaxy fluctuation decomposed in terms of Fourier spherical harmonic modes provide a natural orthogonal basis for all-sky analysis.
- Therefore the spherical Fourier analysis provides a complete and comprehensive description of galaxy clustering and its associated effects.
- Working in spherical Bessel coordinates, it is possible to derive a compact expression for the Power Spectrum and Bispectrum that encompasses all the physical contribution at first and second order, including integrated terms for radial configurations.


Thank You !