Learning about perturbation theory from Numerical Relativity: implications for computing observables.

Tom Giblin
May 29, 2018
CosmoBack, Laboratoire d’Astrophysique de Marseille
Marseille, France

1511.01105, 1511.01106, 1608.04403, to appear

work done with James Mertens
Glenn Starkman, and Chi Tian
The History of Hubble’s Law

- General Relativity
  - F-L-RW say that a static Universe isn’t a solution to GR
  - Give a mathematical description of the relationship between scale factor and energy density
- Einstein Introduces the Cosmological Constant
- Hubble (Lemaitre?) Discovers the expanding Universe
  - This expansion matches the F-L-RW prediction
- Einstein rescinds Cosmological Constant
- Distant Supernova cause us to re-insert the Cosmological Constant (Dark Energy)
The History of Hubble’s Law

- General Relativity
  - F-L-RW say that a static Universe isn’t a solution to GR
  - Give a mathematical description of the relationship between scale factor and energy density
- Einstein Introduces the Cosmological Constant
- Hubble (Lemaitre?) Discovers the expanding Universe
  - This expansion matches the F-L-RW prediction
- Einstein rescinds Cosmological Constant
- Distant Supernova cause us to re-insert the Cosmological Constant (Dark Energy)

This is done under a set of assumptions. Do we understand (or trust these?)
Gravity is Non-Linear

- We like to separate scales when doing physics problems (e.g. what happens here, stays here)

- Non-linear physics can mix up scales - power transferred between scales is often referred to as cascades or inverse-cascades

- The Averaging Problem: When we talk about the expansion of the Universe on the largest of scales, is there any contribution from smaller scales?
Gravity is Non-Linear

- Non-linear parameters were then derived to be degeneracy. The marginalized constraints on the two parameters were mostly constrained through the combination inverse-cascades of last scattering. The measurement of seven acoustic peaks enables one to determine the constraints on non-Gaussianity derived by Planck.

- The results nicely confirm the standard cosmological model of a spatially Euclidean FL universe with a cosmological constant. The recent analysis of the Planck results reveals a tension between the best fits for the Hubble parameter $H_0$ and the matter density parameter $\Omega_m$, defined as the ratio between the Hubble parameter and the matter density parameter, $H_0/\Omega_m$, and the angular distance at the time the sound horizon and the angular distance at the time the constraints on non-Gaussianity derived by Planck.

- Among the constraints derived from the CMB, the results nicely confirm the standard cosmological model of a spatially Euclidean FL universe with a cosmological constant.

- The recent analysis of the Planck results reveals a tension between the best fits for the Hubble parameter $H_0$ and the matter density parameter $\Omega_m$, defined as the ratio between the Hubble parameter and the matter density parameter, $H_0/\Omega_m$, and the angular distance at the time the constraints on non-Gaussianity derived by Planck.

- The results nicely confirm the standard cosmological model of a spatially Euclidean FL universe with a cosmological constant.

- The recent analysis of the Planck results reveals a tension between the best fits for the Hubble parameter $H_0$ and the matter density parameter $\Omega_m$, defined as the ratio between the Hubble parameter and the matter density parameter, $H_0/\Omega_m$, and the angular distance at the time the constraints on non-Gaussianity derived by Planck.

- The recent analysis of the Planck results reveals a tension between the best fits for the Hubble parameter $H_0$ and the matter density parameter $\Omega_m$, defined as the ratio between the Hubble parameter and the matter density parameter, $H_0/\Omega_m$, and the angular distance at the time the constraints on non-Gaussianity derived by Planck.

- The recent analysis of the Planck results reveals a tension between the best fits for the Hubble parameter $H_0$ and the matter density parameter $\Omega_m$, defined as the ratio between the Hubble parameter and the matter density parameter, $H_0/\Omega_m$, and the angular distance at the time the constraints on non-Gaussianity derived by Planck.
Averaging

• Generally a Hubble Volume is taken to be the region over which we do averaging — we all agree that different Hubble patches could have different expansion rates (causality, right?)

\[ H^{-3} \approx (4000 \text{ Mpc})^3 \]

• Yet there is structure at (just) smaller scales
  • Galaxy Clusters \( \sim 1 - 10 \text{ Mpc} \)
  • Inter-Cluster Distances \( \sim 50 \text{ Mpc} \)
Can fully non-linear GR help address these effects?
In[9]:= SetDirectory[NotebookDirectory[]];

In[10]:= << GREAT.m

GREAT functions are: IMetric, Christoffel, Riemann, Ricci, SCurvature, EinsteinTensor, SqRicci, SqRiemann.

Enter 'helpGREAT' for this list of functions

In[11]:= (metric = {{g00[x0, x1, x2, x3], g01[x0, x1, x2, x3], g02[x0, x1, x2, x3], g03[x0, x1, x2, x3]},
{g01[x0, x1, x2, x3], g11[x0, x1, x2, x3], g12[x0, x1, x2, x3], g13[x0, x1, x2, x3]},
{g02[x0, x1, x2, x3], g12[x0, x1, x2, x3], g22[x0, x1, x2, x3]},
{g03[x0, x1, x2, x3], g13[x0, x1, x2, x3], g23[x0, x1, x2, x3], g33[x0, x1, x2, x3]}}) // MatrixForm

Out[11]//MatrixForm=

\[
\begin{pmatrix}
g_{00}(x_0, x_1, x_2, x_3) & g_{01}(x_0, x_1, x_2, x_3) & g_{02}(x_0, x_1, x_2, x_3) & g_{03}(x_0, x_1, x_2, x_3) \\
g_{01}(x_0, x_1, x_2, x_3) & g_{11}(x_0, x_1, x_2, x_3) & g_{12}(x_0, x_1, x_2, x_3) & g_{13}(x_0, x_1, x_2, x_3) \\
g_{02}(x_0, x_1, x_2, x_3) & g_{12}(x_0, x_1, x_2, x_3) & g_{22}(x_0, x_1, x_2, x_3) & \\
g_{03}(x_0, x_1, x_2, x_3) & g_{13}(x_0, x_1, x_2, x_3) & g_{23}(x_0, x_1, x_2, x_3) & g_{33}(x_0, x_1, x_2, x_3)
\end{pmatrix}
\]

In[12]:= coords = {x0, x1, x2, x3}

Out[12] = {x0, x1, x2, x3}

In[13]:= EinsteinTensor[metric, coords]
What can we do?

- You can do a little better by making gauge choices that reduce the number of parameters or (re)parameterize so that you have nice equations for.. some.. of them...

- Even then they are extremely difficult to numerically stabilize
What we have to do...

- Luckily there are a set of new approaches. We use the most common of these: the BSSN formalism.

- It is based on the ADM metric decomposition

\[ g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \gamma_{lk} \beta^l \beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix} \]

- We introduce more parameters than (minimally) necessary so that the equations are easier to solve
In Cosmology

• We can fix the gauge (we will give up being able to create black holes, as well as some other concessions) to focus on spatial slices

• We can then track the spatial 3-metric

\[ \gamma_{ij} = e^{4\phi} \bar{\gamma}_{ij} \]

• as well as the extrinsic curvature

\[ K_{ij} = e^{4\phi} \bar{A}_{ij} + \frac{1}{3} \gamma_{ij} K \]
In Cosmology

• We can fix the gauge (we will give up being able to create black holes, as well as some other concessions) to focus on spatial slices

• We can then track the spatial 3-metric

$$\gamma_{ij} = e^{4\phi} \bar{\gamma}_{ij}$$

• as well as the extrinsic curvature

$$K_{ij} = e^{4\phi} \bar{A}_{ij} + \frac{1}{3} \gamma_{ij} K$$
In Cosmology

- We can fix the gauge (we will give up being able to create black holes, as well as some other concessions) to focus on spatial slices.

- We can then track the spatial 3-metric:

\[ \gamma_{ij} = e^{4\phi} \bar{\gamma}_{ij} \]

Think of this as keeping track of the size of local volumes.

- as well as the extrinsic curvature:

\[ K_{ij} = e^{4\phi} \bar{A}_{ij} + \frac{1}{3} \gamma_{ij} K \]

Think of this as measuring the local expansion rate.
Importantly

These variables have well-behaved differential equations and are a complete description of GR without additional constraints.
Importantly

These variables have well-behaved differential equations and are a complete description of GR without additional constraints.

\[ \partial_t \phi = -\frac{1}{6} K \]

\[ \partial_t \tilde{\gamma}_{ij} = -2 \tilde{A}_{ij} \]

\[ \partial_t K = \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} \]

\[ \partial_t \tilde{A}_{ij} = e^{-4\phi} (R_{ij} - \frac{2}{3} \tilde{\gamma}^{ik} \tilde{\gamma}_{jk} \tilde{A}^{kj}) - \frac{4}{3} \tilde{\gamma}^{ij} \partial_j K - 16\pi \gamma^{ij} S_j + 12 \tilde{A}^{ij} \partial_j \phi. \]

For most of the work here, we chose synchronous gauge (cosmology) / geodesic slicing (Numerical GR)

\[ \alpha = 1, \ \beta^i = 0 \]
With a Source

- As a first-guess; we take a Universe to be filled with a pressureless, non-interacting* perfect fluid with

\[ w = 0 \]

- This fluid obeys a fluid equation,

\[ \partial_t \tilde{D} = \partial_t (\gamma^{1/2} \rho_0) = 0 \]

- which vanishes in synchronous gauge. *Therefore the fluid doesn’t evolve (in our coordinates)
How do we parameterize success?

• Reproducing GR requires the additional satisfaction of a set of constraints

• The Hamiltonian Constraint:

\[ \mathcal{H} \equiv \tilde{\gamma}^{ij} \tilde{D}_i \tilde{D}_j e^\phi - \frac{e^\phi}{8} \tilde{R} + \frac{e^{5\phi}}{8} \tilde{A}_{ij} \tilde{A}^{ij} - \frac{e^{5\phi}}{12} K^2 + 2\pi e^{5\phi} \rho = 0 \]

• The Momentum Constraints:

\[ \mathcal{M}^i = \tilde{D}_j (e^{6\phi} \tilde{A}^{ij}) - \frac{2}{3} e^{6\phi} \tilde{D}^i K - 8\pi e^{10\phi} S^i = 0 \]

• While the BSSN method is analytically equivalent to GR, the numerical implementation can still propagate spurious solutions if you leave the constraint surface
Weren’t you going to talk about physics?

- So we have a numerical framework

- Let’s start a simulation where we have a volume of the Universe with some density perturbations

\[ P_k = \frac{4P_*}{3} \frac{k/k_*}{1 + (k/k_*)^{4/3}} \]

- Then solve the initial condition problem (and put all the inhomogeneities in the volume elements not the expansion rates)
The initial value problem

- By whatever means necessary, we begin with the assumption of homogeneous extrinsic curvature, and the metric response (to the source) is just in the conformal factor,
  \[ \psi \equiv e^\phi \]

- So that the initial conformal factor must obey the following situation,
  \[ \rho = \rho_K + \rho_\psi \]
  \[ \nabla^2 \psi = -2\pi \psi^5 \rho_\psi \]
  \[ K = -\sqrt{24\pi \rho_K} \]
What can we tell about the distribution of $K$?

- We can now compare the statistics of $K$ as a function of the initial density contrast.

- And how that statistic changes in time.

![Graph showing the relationship between $\sigma_p/\bar{\rho}$ and $\bar{\phi}$, with different lines for different values of $\sigma_p/\bar{\rho}$, and a legend indicating the values $0.009$ and $0.107$. The graph also shows a line indicating the evaluated value at $\phi=0.5$.](image-url)
Constructing Null Geodesics

- We start with the geodesic equation
  \[
  \frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}
  \]

- recast in terms of the independent variable (of the code)
  \[
  \frac{d^2 X^\mu}{dt^2} = -\frac{d^2 x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta} \frac{dX^\alpha}{dt} \frac{dX^\beta}{dt} + \Gamma^0_{\alpha\beta} \frac{dX^\mu}{dt} \frac{dX^\alpha}{dt} \frac{dX^\beta}{dt}
  \]

- where we will define
  \[
  q^\mu = \frac{dx^\mu}{dt} = \alpha(n^\mu + V^\mu) \quad \text{and} \quad p^\mu = E(n^\mu + V^\mu)
  \]
Which gives us a set of equations to solve....

\[ \frac{dX^i}{dt} = \alpha V^i - \beta^i \]
\[ \frac{dE}{dt} = E \left( \alpha K_{ij} V^i V^j - V^j \partial_j \alpha \right) \]
\[ \frac{dV^i}{dt} = \alpha V^j \left( \partial_j \ln \alpha - K_{jk} V^k V^i + 2K_j^i - (3) \Gamma_{jk}^i V^k \right) - \gamma^{ij} \partial_j \alpha - V^j \partial_j \beta^i \]

- Which needs to be solved along a set of trajectories
- We don’t know where we end up (only where they start)
- And they don’t lie on lattice points.
No Problem

- We start an large number (500) in arbitrary positions, and in arbitrary directions

- We interpolate the fields along the paths (the lattice points are pretty close together)

- At the end of the simulation we can look at the histories of the particles and draw Hubble Diagrams
Averaged Observers

- **Good News:**
  - Almost indistinguishable agreement with LCDM (and with $\Omega_M = 1$)

- **Bad News:**
  - Only redshift of 0.1...
We look at the residuals

- If we look at the residuals we see that an averaged observer see a matter dominated Hubble diagram
Biased Observer ( kinda )

- The deviations from “straight” aren’t huge for this toy Universe

- So we take a set of points that we know will end up at (approximately) the same location

- Which is an over density of about 10%
And the Residuals are...

We see a bias at low $z$.
But no indications (yet) that this mimics LCDM.
And the Residuals are...

but no indications (yet) that this mimics LCDM
And the Residuals are...
The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways.

\[ \kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A} \]
The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways.

\[ \kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A} \]

- Angular diameter distance in pure FLRW
- True (line-of-sight dependent) angular diameter distance
The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways

\[ \kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A} \]

\[ \kappa = \int (r_s - r) \frac{r}{r_s} \nabla^2_\perp \Phi \, dr \]

\[ \Phi = -\frac{a}{2} \left( 2\dot{a}\dot{B} + a\ddot{B} \right) \]

In a Newtonian treatment
The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways

\[ \kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A} \]

\[ \kappa = \int (r_s - r) \frac{r}{r_s} \nabla_\perp^2 \Phi dr \]

\[ \Phi = -\frac{a}{2} \left( 2\dot{a}\dot{B} + a\ddot{B} \right) \]

\[ D_A = \ell(t_{em})/\varphi(t_{obs}) \]

\[ \frac{d^2}{d\lambda^2} \ell = \ell \left( \mathcal{R} - \sigma^2 \right) \]

direct integration of optical eq.
A single, fully-relativistic simulation allows you to calculate a single observable two ways:

$$\kappa = \int (r_s - r) \frac{r}{r_s} \nabla^2_\perp \Phi \, dr$$

$$\Phi = -\frac{a}{2} \left( 2 \dot{a} \dot{B} + a \ddot{B} \right)$$

$$D_A = \ell(t_{em})/\varphi(t_{obs})$$

$$\frac{d^2}{d\lambda^2} \ell = \ell \left( \mathcal{R} - \sigma^2 \right)$$
The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways

\[ \kappa = \int (r_s - r) \frac{r}{r_s} \nabla^2 \Phi dr \]
\[ \Phi = -\frac{a}{2} \left( 2\dot{a}B + a\ddot{B} \right) \]
\[ D_A = \ell(t_{em})/\varphi(t_{obs}) \]
\[ \frac{d^2}{d\lambda^2} \ell = \ell (R - \sigma^2) \]
The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways

\[ \kappa = \int (r_s - r) \frac{r}{r_s} \nabla^2 \Phi dr \]

\[ \Phi = -\frac{a}{2} \left( 2a \dot{B} + a \ddot{B} \right) \]

\[ D_A = \ell(t_{em})/\varphi(t_{obs}) \]

\[ \frac{d^2}{d\lambda^2} \ell = \ell \left( R - \sigma^2 \right) \]
The effect on the observable correction effects are larger on smaller scales
The Weak Lensing Power Spectrum

In the limit in which you trust linear perturbations, you can define the normal, gauge-independent quantities. From gauge-gauge these quantities agree "well".

\[ ds^2 = -(1 + 2\Phi)dt^2 + 2a(t)B_i dx^i dt + a^2(t) \left[ (1 - 2\Psi)\delta_{ij} + 2\partial_i \partial_j E \right] dx^i dx^j \]

\[ \Phi_B \equiv \Phi - \frac{d}{dt} \left[ a^2 \left( \dot{E} - \frac{B}{a} \right) \right] \quad \Psi_B \equiv \Psi + Ha^2 \left( \dot{E} - \frac{B}{a} \right) \]
To what degree do we see departure from first-order perturbation theory?

Any second-order perturbation theory is gauge-dependent.
The linearized Einstein Equation (asking for a friend)

\[ ds^2 = -(1 + 2\Phi)dt^2 + 2a(t)B_i dx^i dt + a^2(t) [(1 - 2\Psi)\delta_{ij} + 2\partial_i \partial_j E] dx^i dx^j \]

- when you linearize the full Einstein Equations you end up with a set of constraints, e.g.

\[ G \equiv 8\pi Ga^2 \pi s + \Phi + \Psi - a^2 \ddot{E} - 3a\dot{a}E + 2a\dot{B} + 4\dot{a}B = 0 \]

where \[ \delta T^i_i = \delta_{ij} \delta p + \partial_i \partial_j \pi^s \]

- here we’re writing it in terms of the scalar modes only
Violation of the linearized Einstein Equation

\[ \mathcal{G} \equiv 8\pi G a^2 \pi^s + \Phi + \Psi - a^2 \ddot{E} - 3a\dot{a}E + 2a\dot{B} + 4\dot{a}B = 0 \]
Another example

\[ \langle \text{det } \gamma^{1/6} \rangle - \alpha_{\text{FLRW}} \]

\[ \langle \text{det } \gamma^{1/6} \rangle \]

synchronous

harmonic
Another example

Moral: gauge-dependent quantities differ and second-order effects emerge
There’s no indication that this has any effect on observables, however.
Your Take-home

- First-Order perturbation theory has a gauge-independent formalism.
- Gauge-independent parameters agree well in different gauges/slicing.
- Corrections to these parameters are gauge-dependent and look like they change things (but don’t yet have observable consequences).
Fin