Learning about perturbation theory from Numerical Relativity: implications for computing observables.

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May 29, 2018
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1511.01105, 1511.01106,
1608.04403, to appear

work done with James Mertens
Glenn Starkman, and Chi Tian

The History of Hubble's Law

- General Relativity
  - F-L-RW say that a static Universe isn't a solution to GR
  - Give a mathematical description of the relationship between scale factor and energy density
- Einstein Introduces the Cosmological Constant
- Hubble (Lemaitre?) Discovers the expanding Universe
  - This expansion matches the F-L-RW *prediction*
- Einstein rescinds Cosmological Constant
- Distant Supernova cause us to re-insert the Cosmological Constant (Dark Energy)

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This is done under a set of assumptions. Do we understand (or trust these?)

Gravity is Non-Linear

- We like to *separate scales* when doing physics problems (e.g. what happens here, stays here)
- Non-linear physics can mix up scales - power transferred between scales is often referred to as *cascades* or *inverse-cascades*
- The Averaging Problem : When we talk about the expansion of the Universe on the largest of scales, is there *any contribution* from smaller scales?

Gravity is Non-Linear

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  transferred b
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- The Averagi
  expansion o
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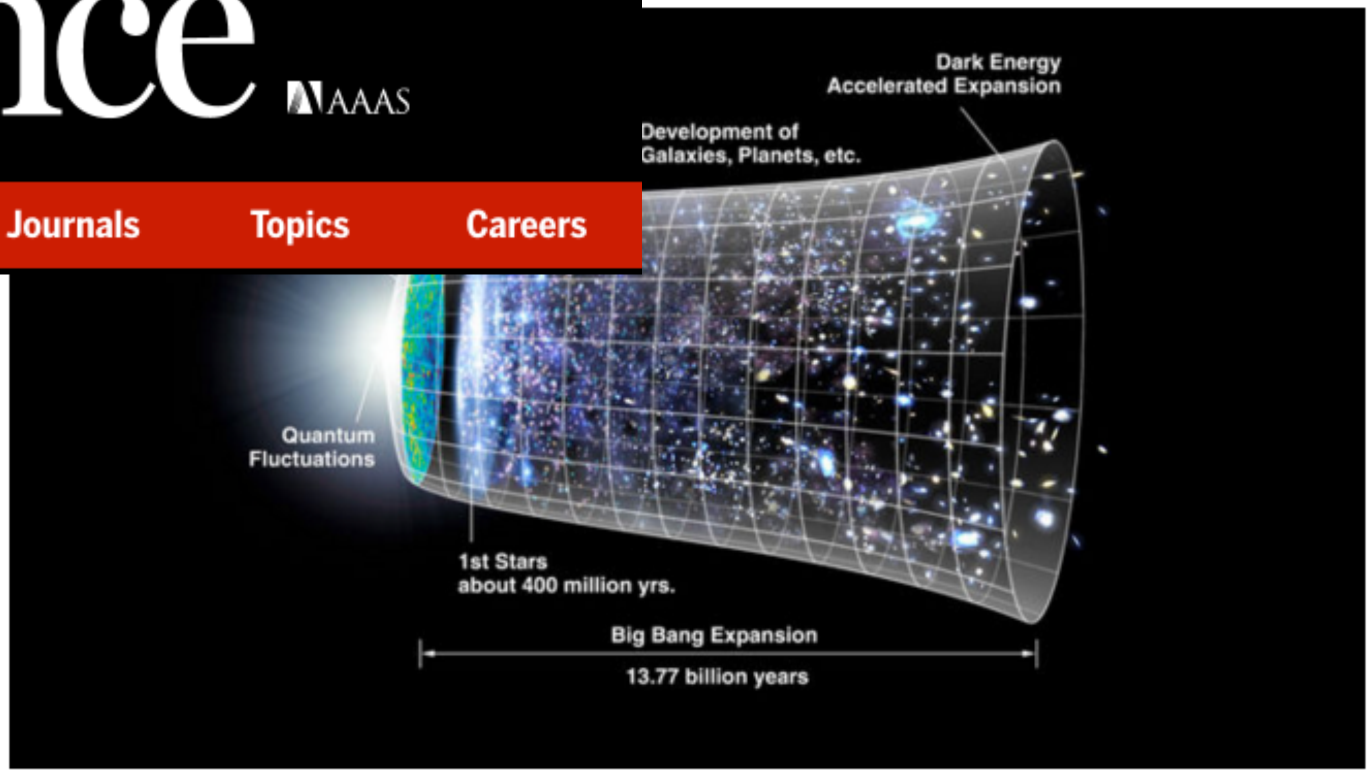


Long after the big bang, the expansion of the universe has again begun to accelerate, as shown in this diagram. NASA/WMAP Science Team

Is dark energy an illusion?

By Adrian Cho | Apr. 3, 2017 , 2:00 PM

Averaging

- Generally a Hubble Volume is taken to be the region over which we do averaging — we all agree that different Hubble patches could have different expansion rates (causality, right?)

$$H^{-3} \approx (4000\,\mathrm{Mpc})^3$$

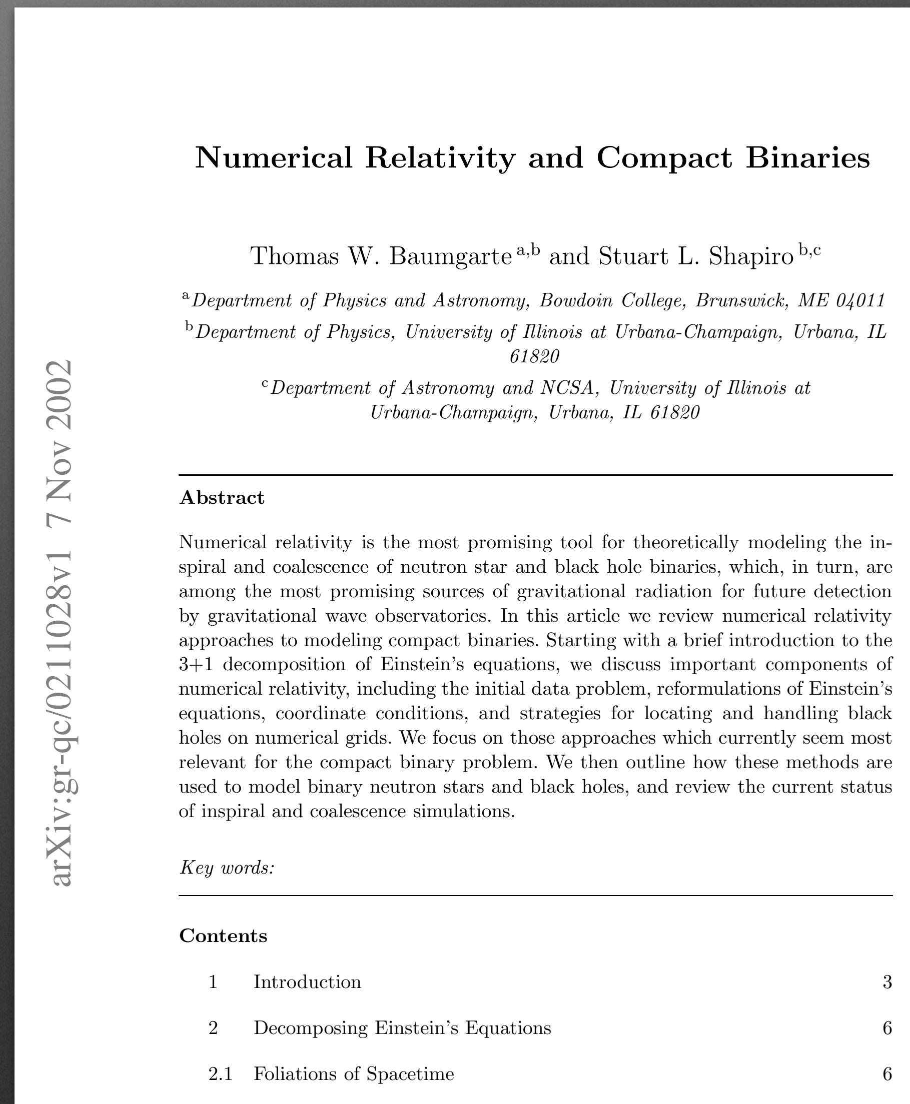
- Yet there is structure at (just) smaller scales
  - Galaxy Clusters $\sim 1 - 10\,\mathrm{Mpc}$
  - Inter-Cluster Distances $\sim 50\,\mathrm{Mpc}$

Can fully non-linear GR help address these effects?

```
In[9]:= SetDirectory[NotebookDirectory[]];

In[10]:= << GREAT.m

GREAT functions are: IMetric, Christoffel,
  Riemann, Ricci, SCurvature, EinsteinTensor, SqRicci, SqRiemann.

Enter 'helpGREAT' for this list of functions

In[11]:= (metric = {{g00[x0, x1, x2, x3], g01[x0, x1, x2, x3], g02[x0, x1, x2, x3],
       g03[x0, x1, x2, x3]}, {g01[x0, x1, x2, x3], g11[x0, x1, x2, x3],
       g12[x0, x1, x2, x3], g03[x0, x1, x2, x3]},
      {g02[x0, x1, x2, x3], g12[x0, x1, x2, x3], g22[x0, x1, x2, x3],
       g23[x0, x1, x2, x3]}, {g03[x0, x1, x2, x3], g13[x0, x1, x2, x3],
       g23[x0, x1, x2, x3], g33[x0, x1, x2, x3]}}) // MatrixForm

Out[11]//MatrixForm=
 g00[x0, x1, x2, x3]  g01[x0, x1, x2, x3]  g02[x0, x1, x2, x3]  g03[x0, x1, x2, x3]
 g01[x0, x1, x2, x3]  g11[x0, x1, x2, x3]  g12[x0, x1, x2, x3]  g03[x0, x1, x2, x3]
 g02[x0, x1, x2, x3]  g12[x0, x1, x2, x3]  g22[x0, x1, x2, x3]  g23[x0, x1, x2, x3]
 g03[x0, x1, x2, x3]  g13[x0, x1, x2, x3]  g23[x0, x1, x2, x3]  g33[x0, x1, x2, x3]

In[12]:= coords = {x0, x1, x2, x3}

Out[12]= {x0, x1, x2, x3}

In[13]:= EinsteinTensor[metric, coords]
```

What can we do?

- You can do a little better by making gauge choices that reduce the number of parameters or (re)parameterize so that you have nice equations for.. some.. of them…
- Even then they are extremely difficult to numerically stabilize



Numerical Relativity and Compact Binaries

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Abstract

Numerical relativity is the most promising tool for theoretically modeling the inspiral and coalescence of neutron star and black hole binaries, which, in turn, are among the most promising sources of gravitational radiation for future detection by gravitational wave observatories. In this article we review numerical relativity approaches to modeling compact binaries. Starting with a brief introduction to the 3+1 decomposition of Einstein's equations, we discuss important components of numerical relativity, including the initial data problem, reformulations of Einstein's equations, coordinate conditions, and strategies for locating and handling black holes on numerical grids. We focus on those approaches which currently seem most relevant for the compact binary problem. We then outline how these methods are used to model binary neutron stars and black holes, and review the current status of inspiral and coalescence simulations.

Key words:

Contents

What we have to do…

- Luckily there are a set of new approaches. We use the most common of these: the BSSN formalism.
- It is based on the ADM metric decomposition

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \gamma_{lk}\beta^l\beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}$$

- We we introduce more parameters than (minimally) necessary so that the equations are easier to solve

In Cosmology

- We can fix the gauge (we will give up being able to create black holes, as well as some other concessions) to focus on spatial slices

- We can then track the spatial 3-metric

$$\gamma_{ij} = e^{4\phi}\bar{\gamma}_{ij}$$

- as well as the extrinsic curvature

$$K_{ij} = e^{4\phi}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

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In Cosmology

- We can fix the gauge (we will give up being able to create black holes, as well as some other concessions) to focus on spatial slices

- We can then track the spatial 3-metric

Think of this as keeping track of the size of local volumes

$$\gamma_{ij} = e^{4\phi}\bar{\gamma}_{ij}$$

- as well as the extrinsic curvature

Think of this as measuring the local expansion rate

$$K_{ij} = e^{4\phi}\bar{A}_{ij} + \frac{1}{3}\gamma_{ij}K$$

Importantly

These variables have well-behaved differential equations and *are a complete description of GR without additional constraints*

$$\partial_t \phi = -\frac{1}{6} K$$

$$\partial_t \bar{\gamma}_{ij} = -2 \bar{A}_{ij}$$

$$\partial_t K = \bar{A}_{ij} \bar{A}^{ij} + \frac{1}{3} K^2 + 4\pi(\rho + S)$$

$$\partial_t \bar{A}_{ij} = e^{-4\phi} (R_{ij} - 8\pi S_{ij})^{TF} + K \bar{A}_{ij} - 2 \bar{A}_{il} \bar{A}^l_j$$

$$\partial_t \bar{\Gamma}^i = 2 \bar{\Gamma}^i_{jk} \bar{A}^{jk} - \frac{4}{3} \bar{\gamma}^{ij} \partial_j K - 16\pi \bar{\gamma}^{ij} S_j + 12 \bar{A}^{ij} \partial_j \phi \, .$$

Importantly

These variables have well-behaved differential equations and *are a complete description* ... *tional* ... *raints*

$$\partial_t \phi = -\frac{1}{6} K$$

$$\partial_t \bar{\gamma}_{ij} = -2\bar{A}_{ij}$$

$$\partial_t K = \bar{A}_{ij}\bar{A}^{ij} + \frac{1}{3} \ldots$$

$$\partial_t \bar{A}_{ij} = e^{-4\phi}(R_{ij} - \ldots$$

$$\partial_t \bar{\Gamma}^i = 2\bar{\Gamma}^i_{jk}\bar{A}^{jk} - \frac{4}{3}\bar{\gamma}^{ij}\partial_j K - 16\pi\bar{\gamma}^{ij}S_j + 12\bar{A}^{ij}\partial_j \phi\,.$$

For most of the work here, we chose synchronous gauge (cosmology) / geodesic slicing (Numerical GR)

$$\alpha = 1,\ \beta^i = 0$$

With a Source

- As a first-guess; we take a Universe to be filled with a pressureless, non-interacting* perfect fluid with

$$w = 0$$

- This fluid obeys a fluid equation,

$$\partial_t \tilde{D} = \partial_t(\gamma^{1/2}\rho_0) = 0$$

- which vanishes in *synchronous* gauge. *Therefore the the fluid doesn't evolve (in our coordinates)

How do we parameterize success?

- Reproducing GR requires the additional satisfaction of a set of constraints
  - The Hamiltonian Constraint:

$$\mathcal{H} \equiv \bar{\gamma}^{ij}\bar{D}_i\bar{D}_j e^{\phi} - \frac{e^{\phi}}{8}\bar{R} + \frac{e^{5\phi}}{8}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{e^{5\phi}}{12}K^2 + 2\pi e^{5\phi}\rho = 0$$

  - The Momentum Constraints:

$$\mathcal{M}^i = \bar{D}_j(e^{6\phi}\tilde{A}^{ij}) - \frac{2}{3}e^{6\phi}\bar{D}^i K - 8\pi e^{10\phi}S^i = 0$$

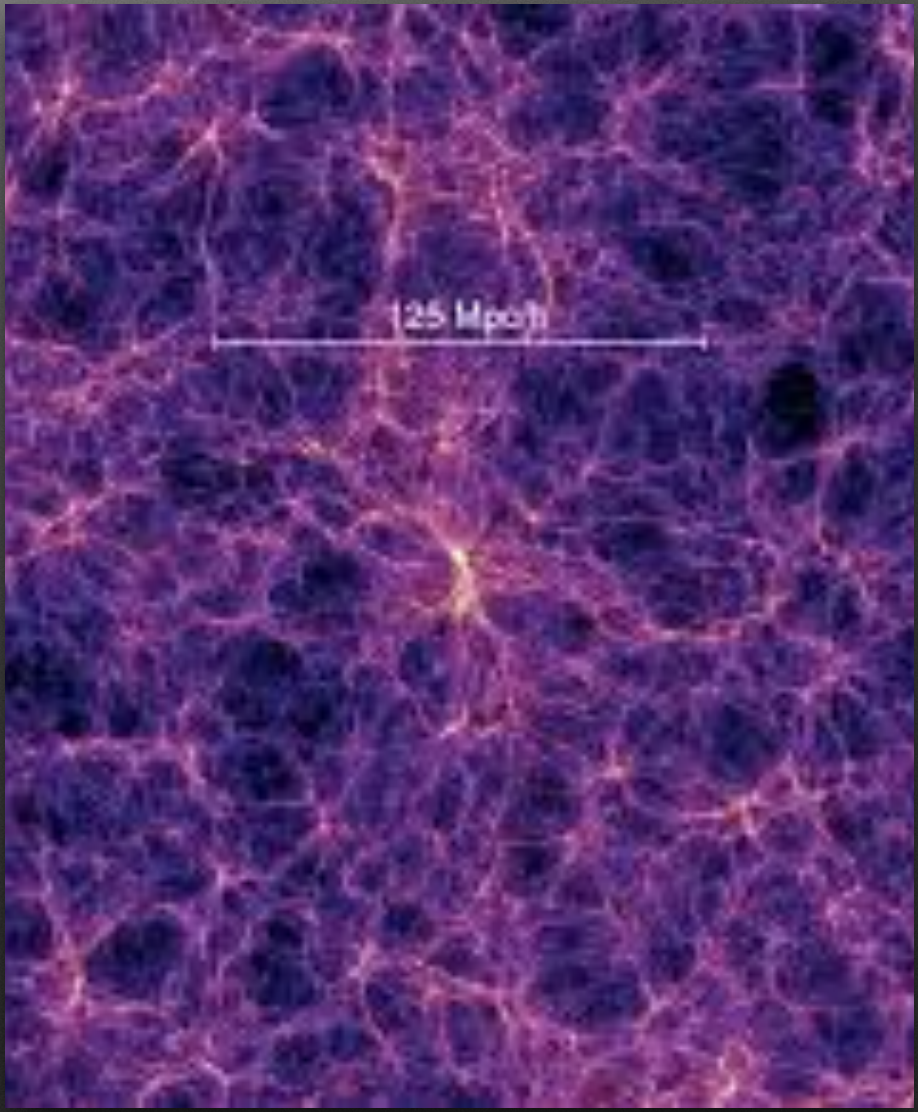
- While the BSSN method is analytically equivalent to GR, the numerical implementation can still propagate spurious solutions if you leave the constraint surface

Weren't you going to talk about physics?

- So we have a numerical framework
- Let's start a simulation where we have a volume of the Universe with some density perturbations

$$P_k = \frac{4P_*}{3}\frac{k/k_*}{1+(k/k_*)^{4/3}}$$

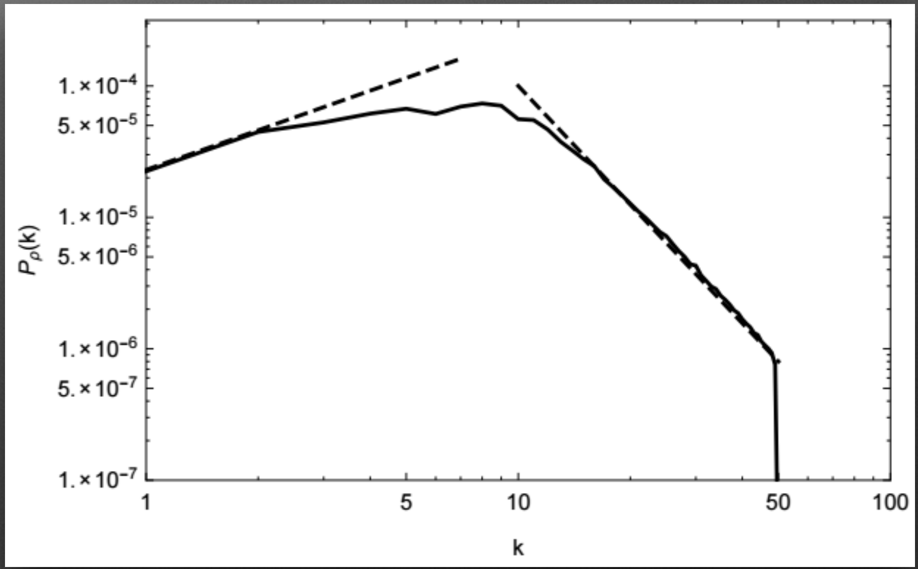
- Then solve the initial condition problem (and put all the inhomogeneities in the volume elements *not* the expansion rates)



The initial value problem

- By whatever means necessary, we begin with the assumption of homogeneous extrinsic curvature, and the metric response (to the source) is just in the conformal factor,

$$\psi \equiv e^{\phi}$$



- So that the initial conformal factor must obey the following situation,

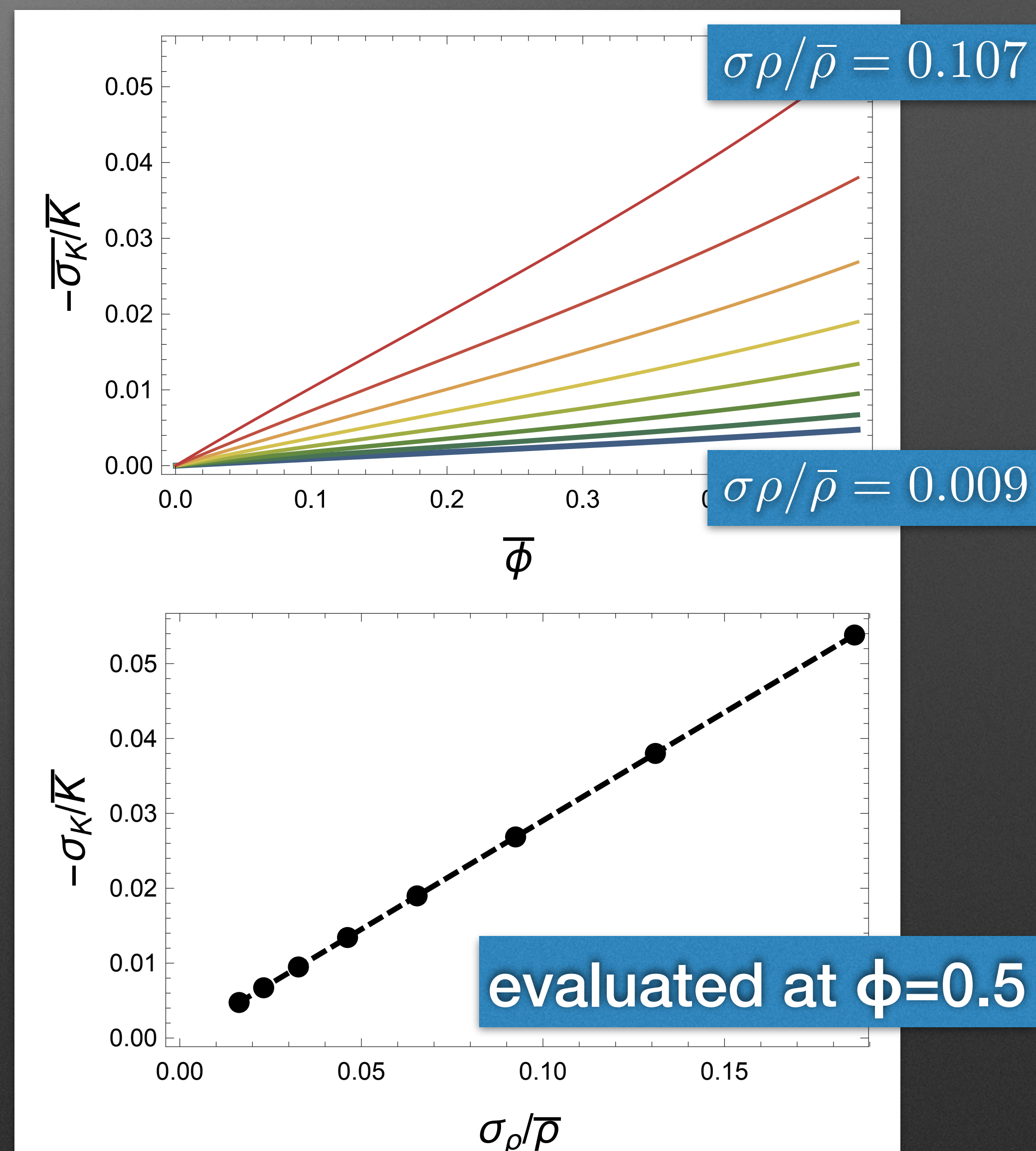
$$\rho = \rho_K + \rho_\psi$$

$$\nabla^2\psi = -2\pi\psi^5\rho_\psi$$

$$K = -\sqrt{24\pi\rho_K}$$

What can we tell about the distribution of *K*?

- We can now compare the statistics of *K* as a function of the initial density contrast
- And how that statistic changes in time



Constructing Null Geodesics

- We start with the geodesic equation

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

- recast in terms of the independent variable (of the code)

$$\frac{\mathrm{d}^2X^\mu}{\mathrm{d}t^2} = -\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}X^\alpha}{\mathrm{d}t}\frac{\mathrm{d}X^\beta}{\mathrm{d}t} + \Gamma^0_{\alpha\beta}\frac{\mathrm{d}X^\alpha}{\mathrm{d}t}\frac{\mathrm{d}X^\beta}{\mathrm{d}t}\frac{\mathrm{d}X^\mu}{\mathrm{d}t}$$

- where we will define

$$q^\mu = \frac{dx^\mu}{dt} = \alpha(n^\mu + V^\mu) \quad \text{and} \quad p^\mu = E(n^\mu + V^\mu)$$

Which gives us a set of equations to solve....

$$\frac{\mathrm{d}X^i}{\mathrm{d}t} = \alpha V^i - \beta^i$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = E\left(\alpha K_{ij} V^i V^j - V^j \partial_j \alpha\right)$$

$$\frac{\mathrm{d}V^i}{\mathrm{d}t} = \quad \alpha V^j \left(V^i \partial_j \ln\alpha - K_{jk} V^k V^i + 2K^i_j - {}^{(3)}\Gamma^i_{jk} V^k\right) - \gamma^{ij}\partial_j \alpha - V^j \partial_j \beta^i$$

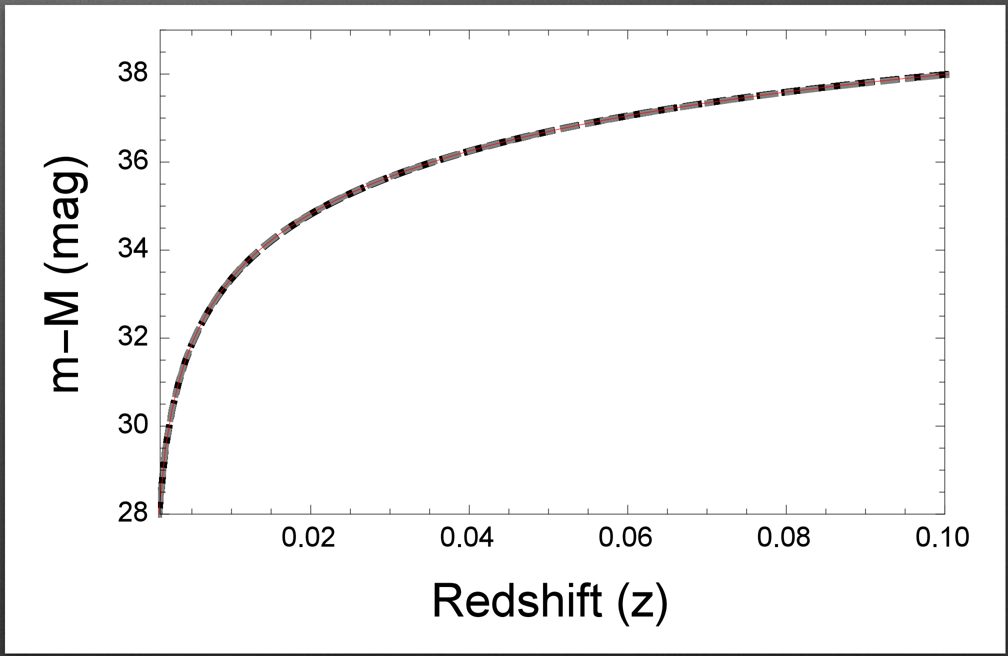
- Which needs to be solved along a set of trajectories
- We don’t know where we end up (only where they start)
- And they don’t lie *on* lattice points.

No Problem

- We start an large number (500) in arbitrary positions, and in arbitrary directions
- We interpolate the fields along the paths (the lattice points are pretty close together)
- At the end of the simulation we can look at the histories of the particles and draw Hubble Diagrams

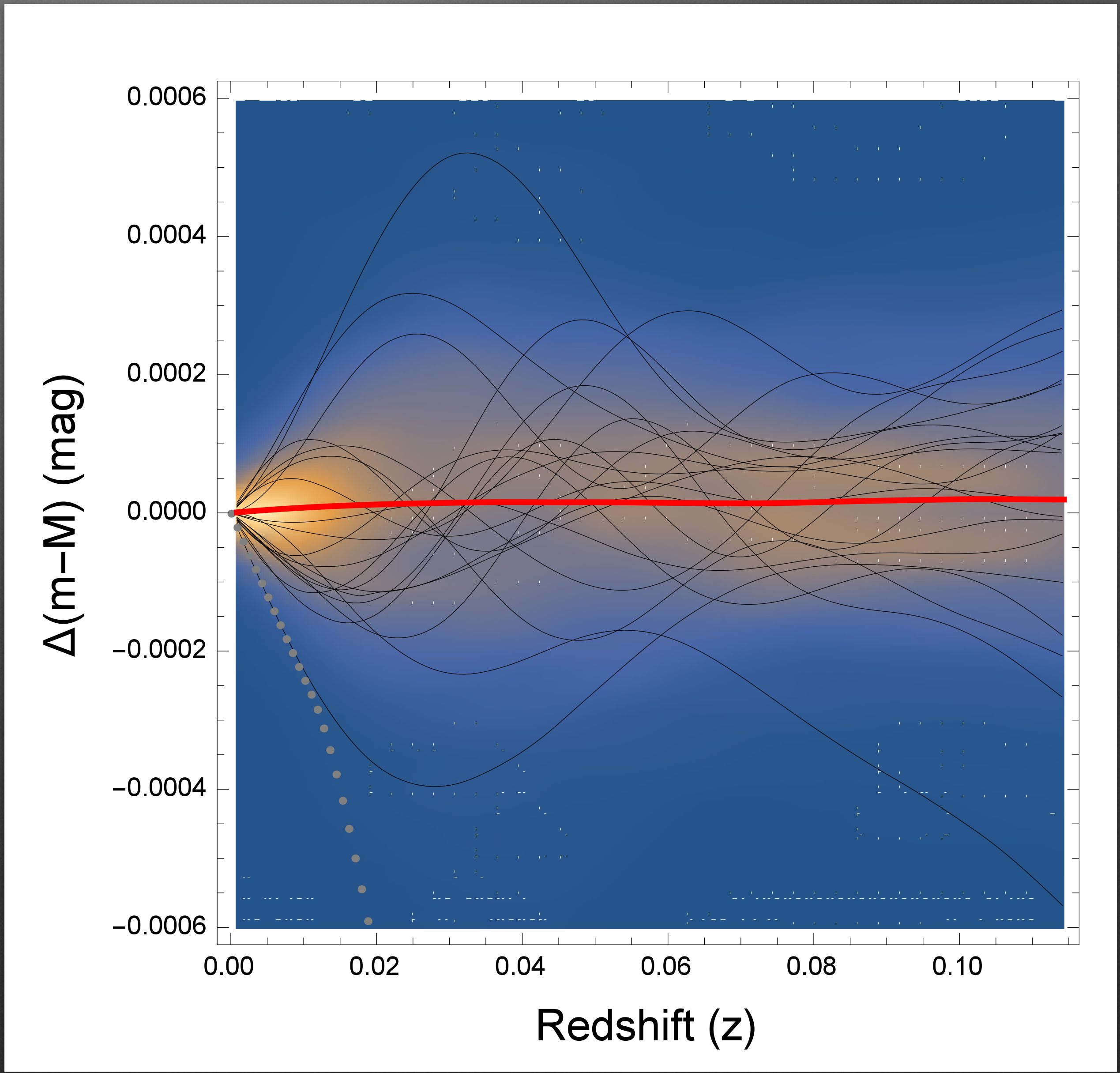
Averaged Observers

- Good News:
  - Almost indistinguishable agreement with LCDM (and with ΩM=1)
- Bad News:
  - Only redshift of 0.1…



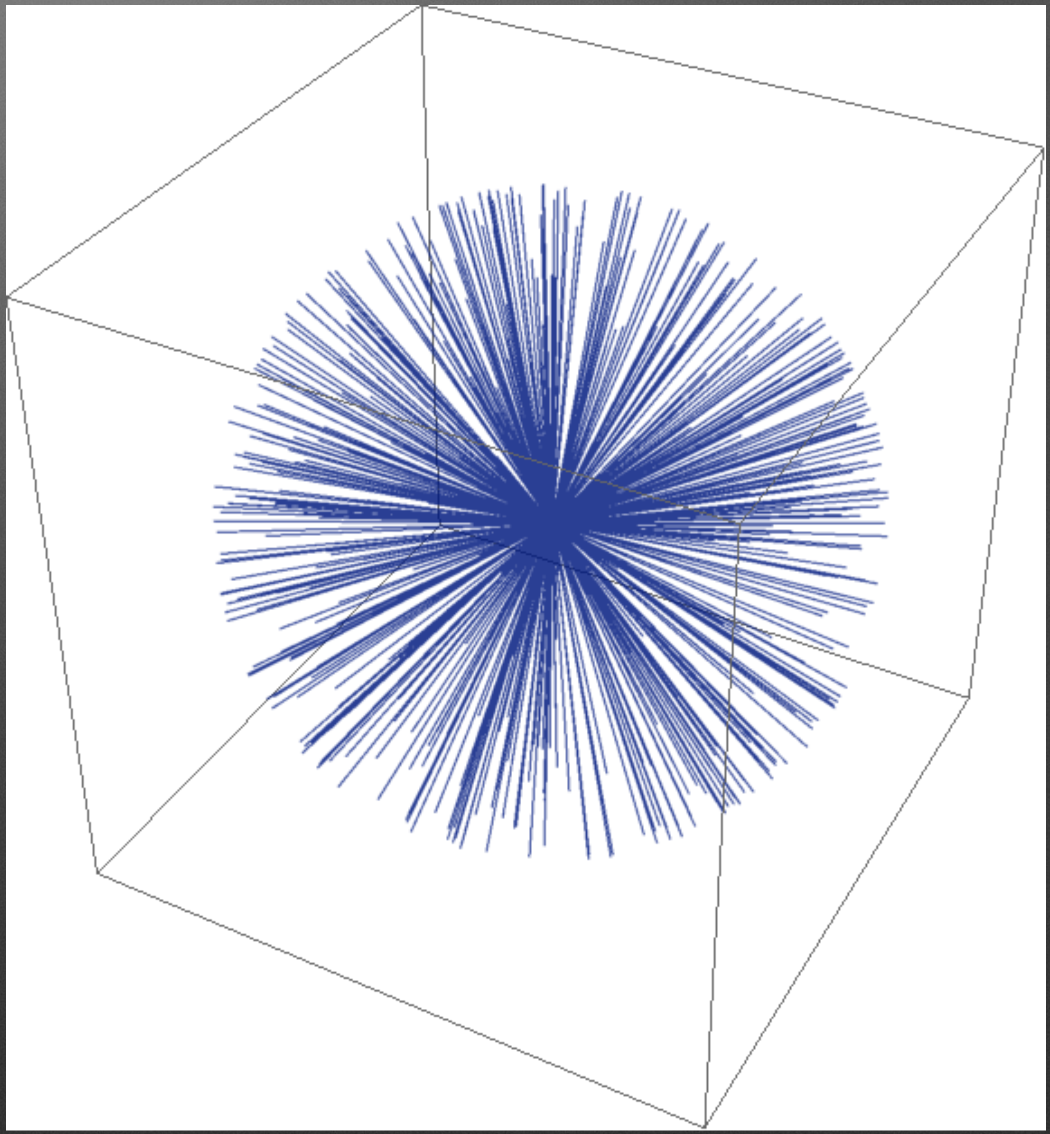
We look at the residuals

- If we look at the residuals we see that an *averaged* observer see a matter dominated Hubble diagram



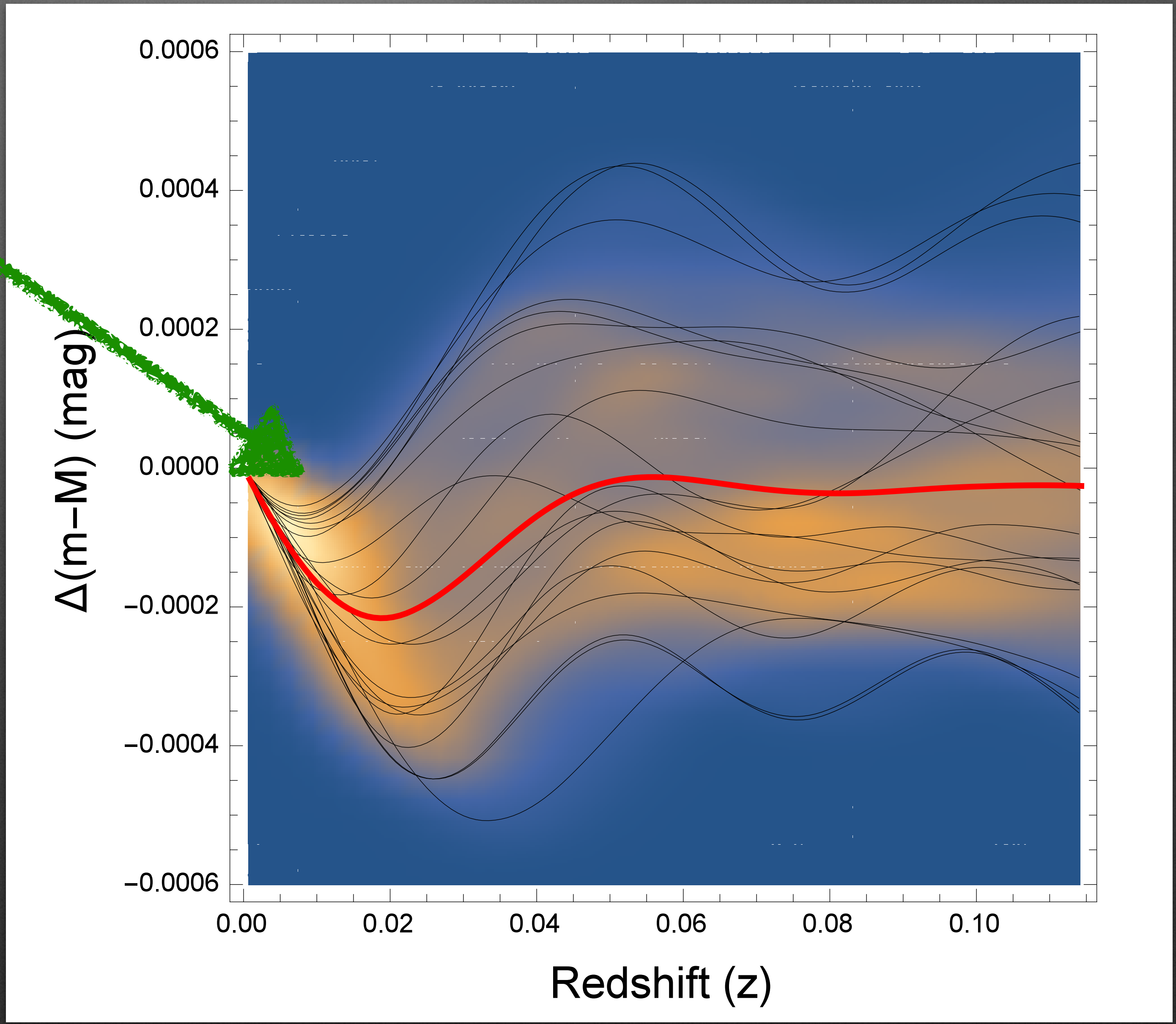
Biased Observer (kinda)

- The deviations from "straight" aren't huge for this toy Universe
- So we take a set of points that we know will end up at (approximately) the same location
- Which is an over density of about 10%



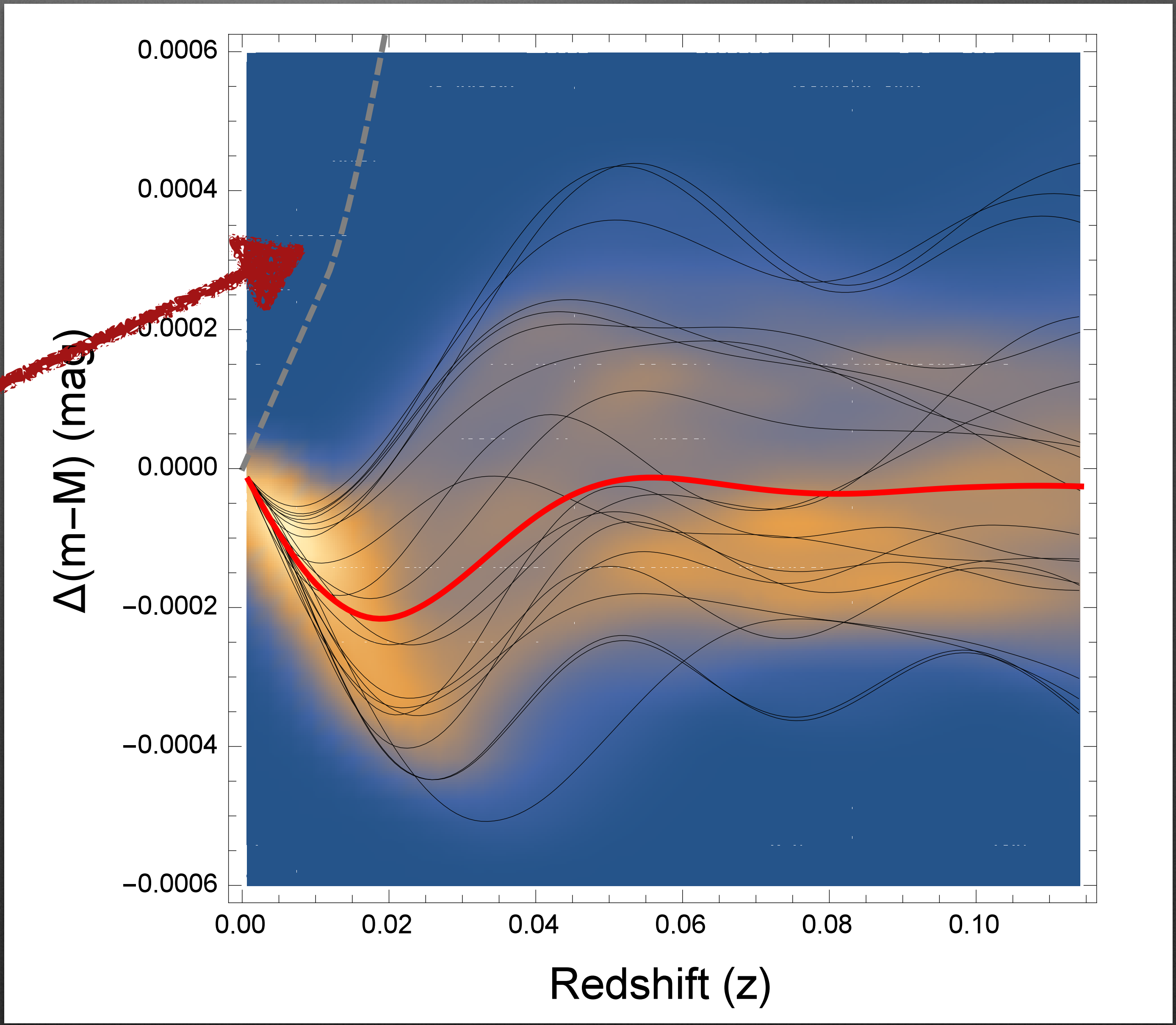
And the Residuals are…

We see
a bias at
low z



And the Residuals are…

but no indications (yet) that this mimics LCDM



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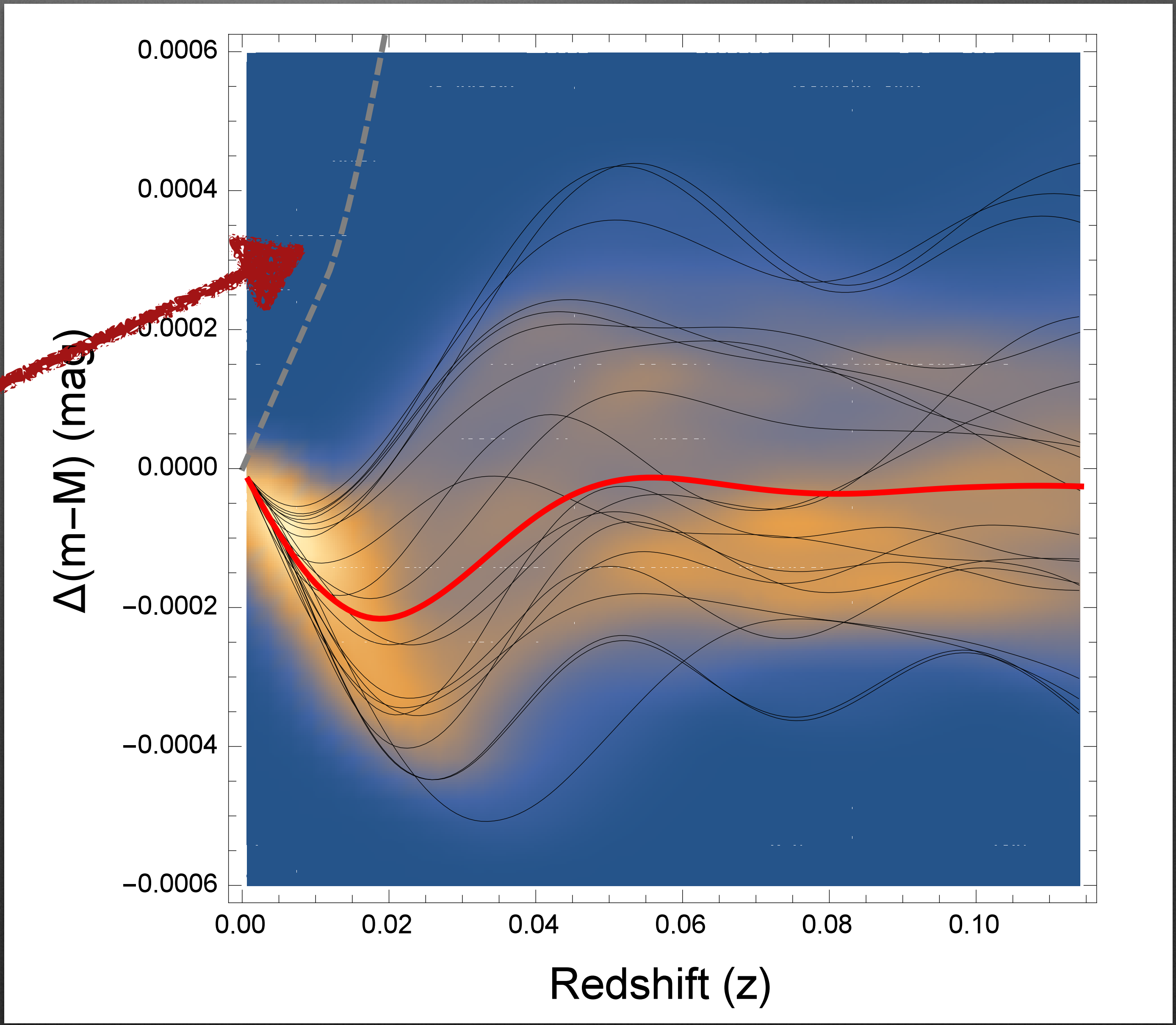

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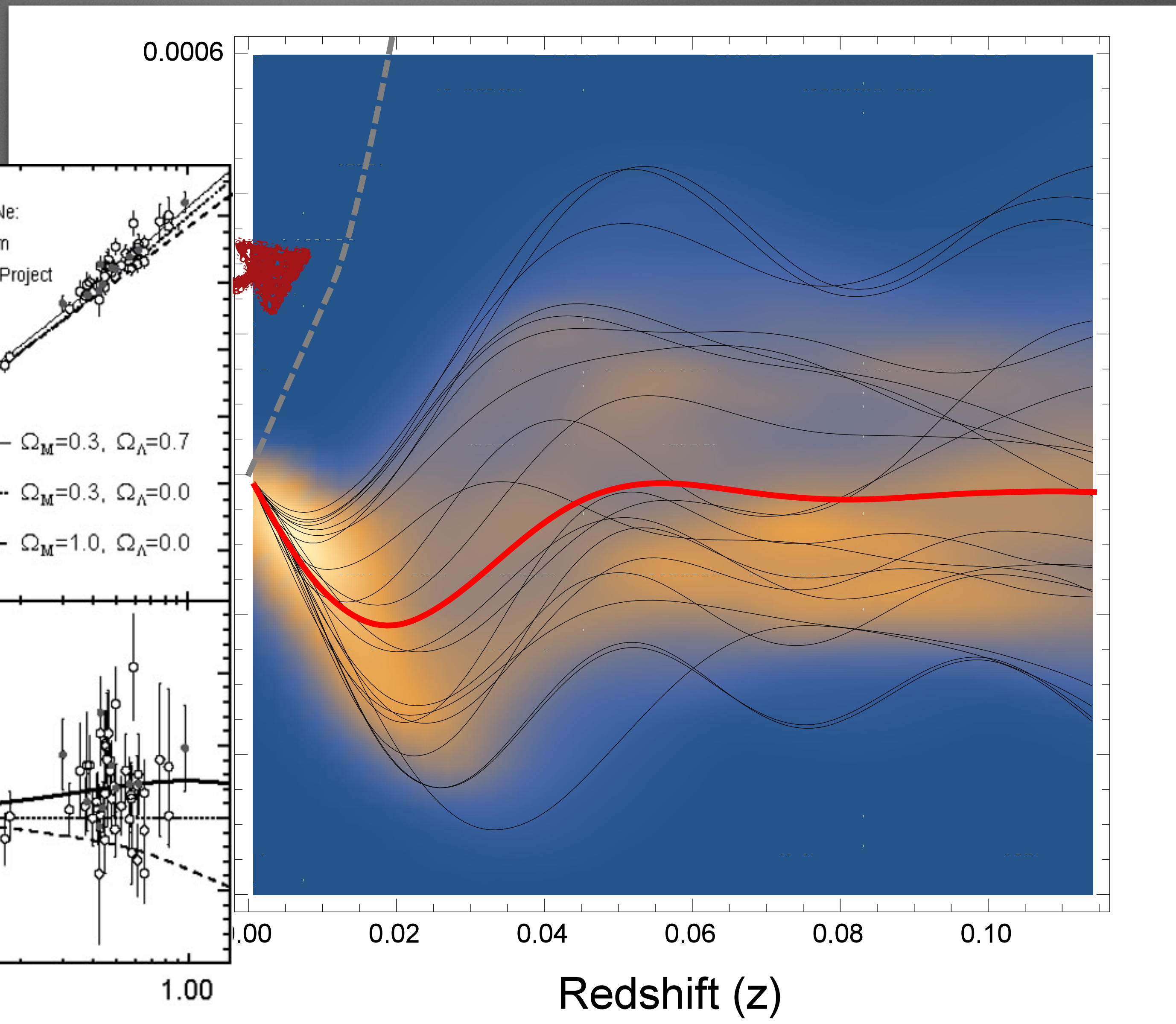

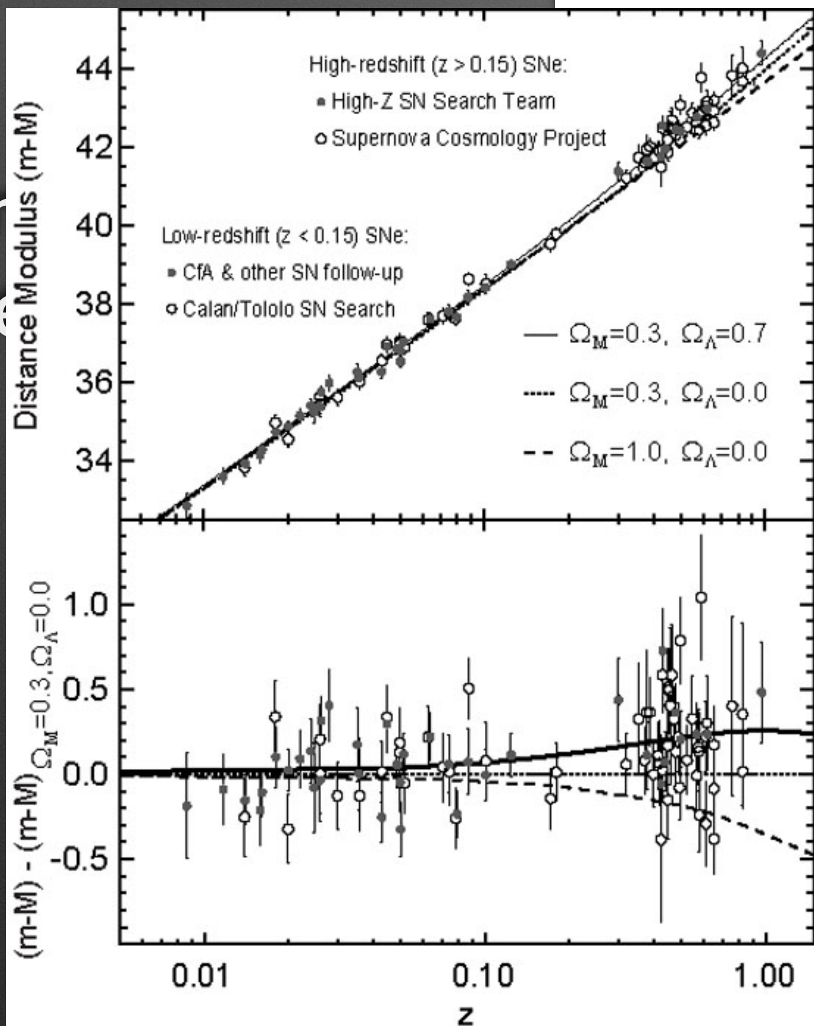

The Weak Lensing Power Spectrum

- A single, fully-relativistic simulation allows you to calculate a single observable two ways

$$\kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A}$$

The Weak Lensing Power Spectrum

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angular diameter distance in pure FLRW

$$\kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A}$$

true (line-of-sight dependent) angular diameter distance

The Weak Lensing Power Spectrum

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$$\kappa \equiv \frac{\bar{D}_A - D_A}{\bar{D}_A}$$

$$\kappa = \int (r_s - r)\, \frac{r}{r_s} \nabla_\perp^2 \Phi dr$$

$$\Phi = -\frac{a}{2}\left(2\dot{a}\dot{B} + a\ddot{B}\right)$$

In a Newtonian treatment

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$$D_A = \ell(t_{\rm em})/\varphi(t_{\rm obs})$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\ell = \ell\left(\mathcal{R} - \sigma^2\right)$$

direct integration of optical eq.

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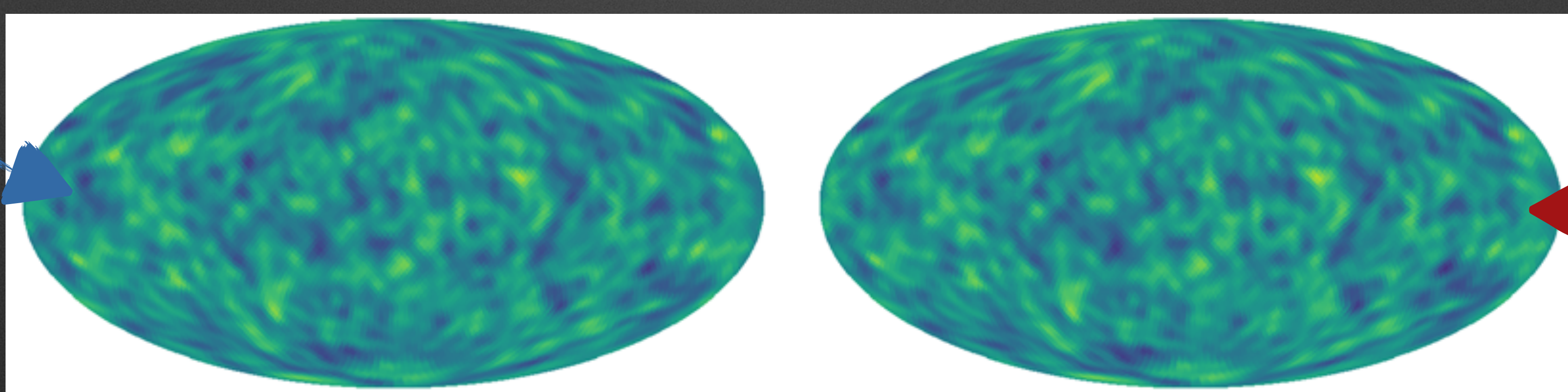
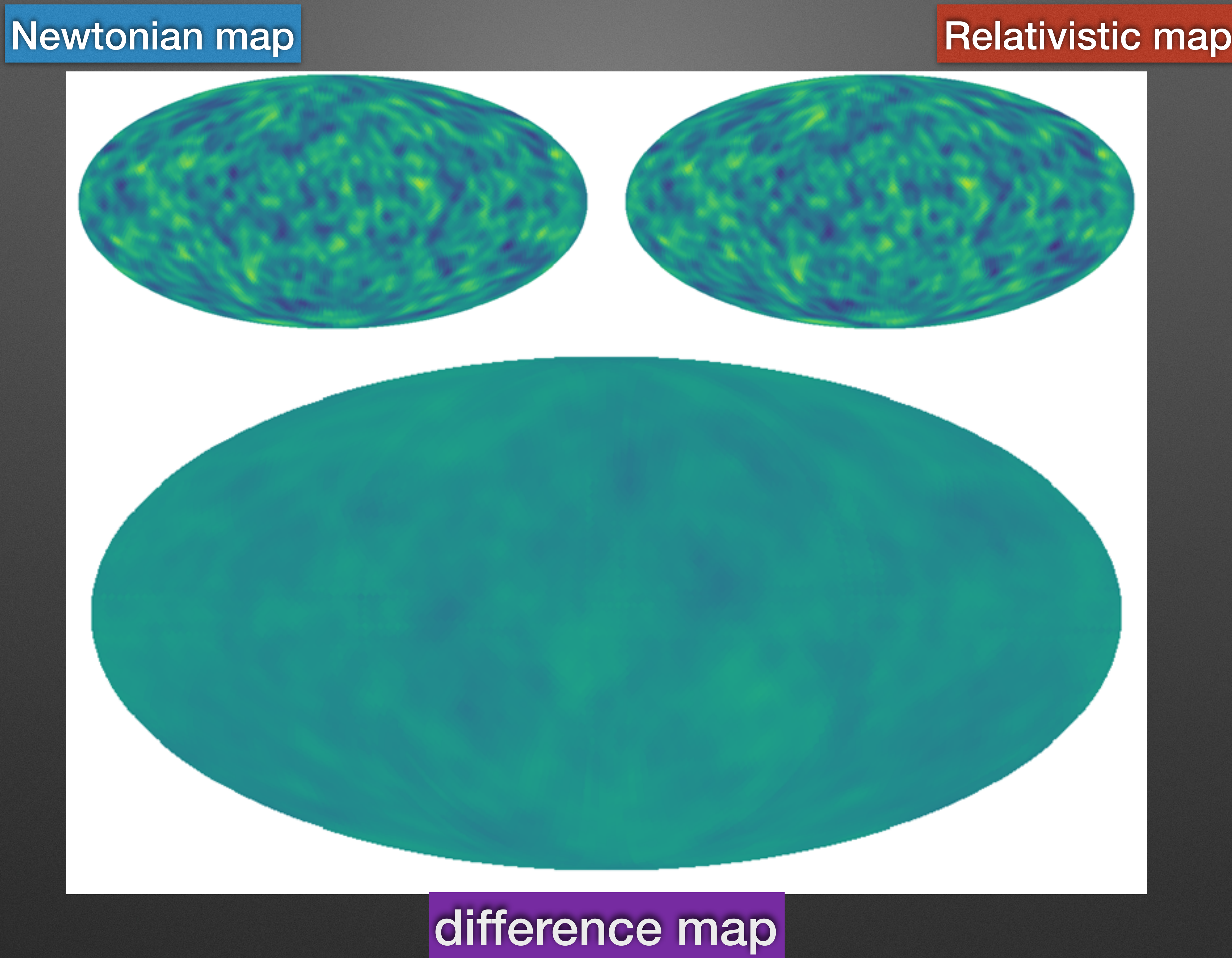
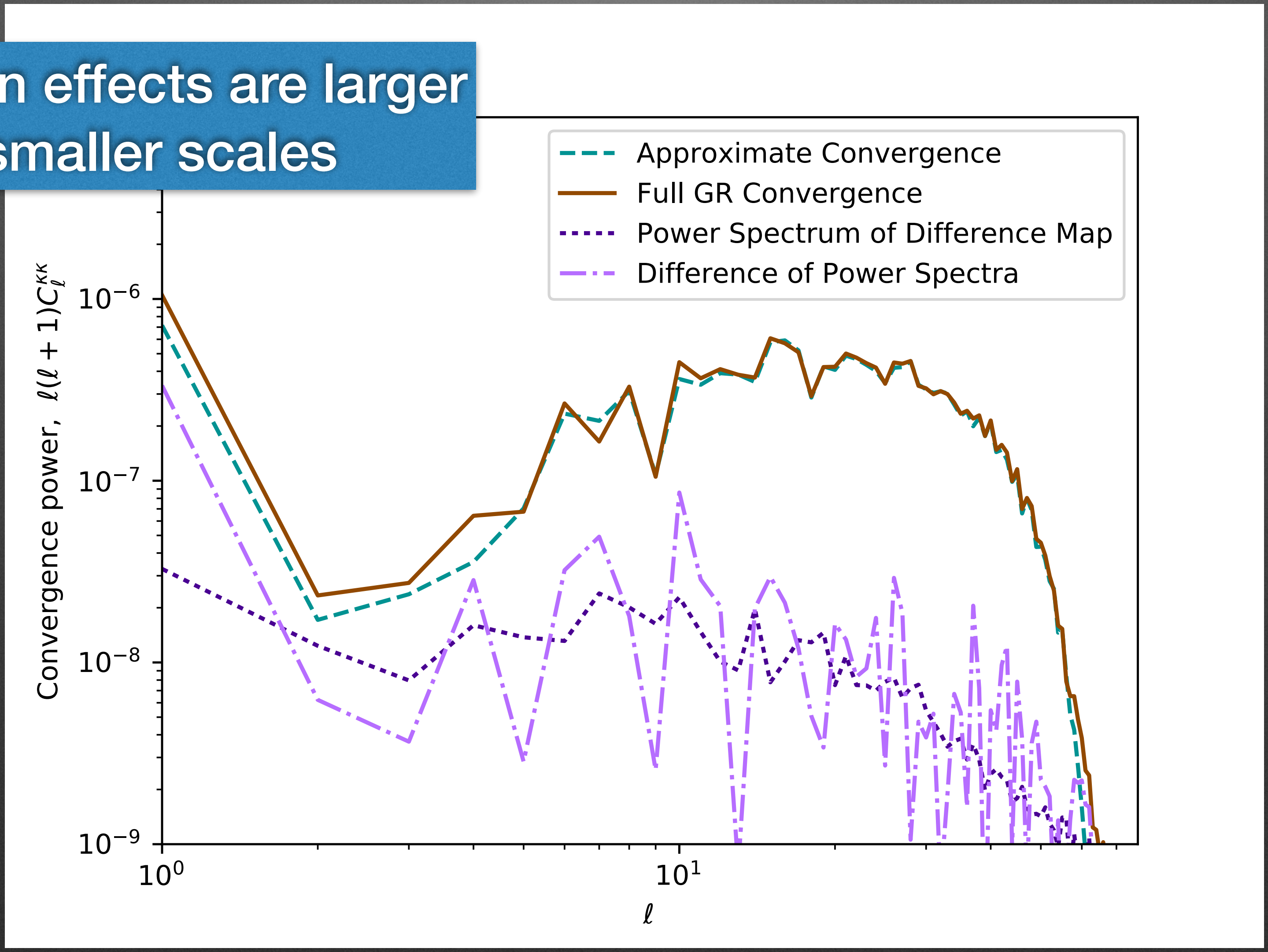



Newtonian map
Relativistic map
difference map

The effect on the observable

correction effects are larger on smaller scales



The Weak Lensing Power Spectrum

- In the limit in which you trust *linear* perturbations

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)B_{,i}dx^i dt \\ +a^2(t)\left[(1-2\Psi)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j$$

- you can define the normal, gauge-independent quantities

$$\Phi_B \equiv \Phi - \frac{d}{dt}\left[a^2\left(\dot{E} - \frac{B}{a}\right)\right] \qquad \Psi_B \equiv \Psi + Ha^2\left(\dot{E} - \frac{B}{a}\right)$$

- from gauge-gauge these quantities agree “well"

To what degree do we see departure from first-order perturbation theory?

Any second-order perturbation theory is gauge-dependent.

The linearized Einstein Equation (asking for a friend)

$$ds^2 = -(1+2\Phi)dt^2 + 2a(t)B_{,i}dx^i dt$$
$$+a^2(t)\left[(1-2\Psi)\delta_{ij} + 2\partial_i\partial_j E\right]dx^i dx^j$$

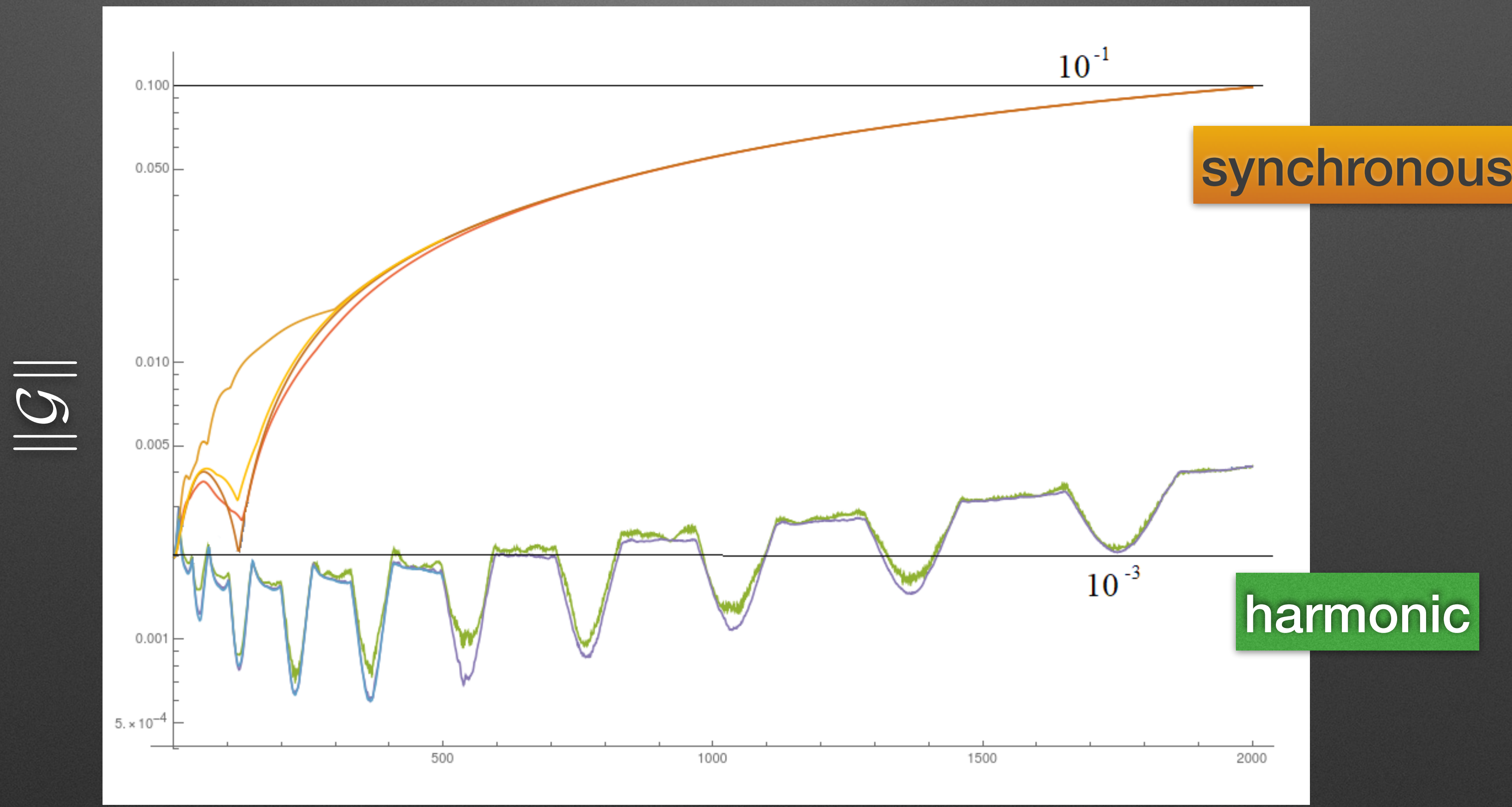
- when you *linearize* the full Einstein Equations you end up with a set of constraints, e.g.

$$\mathcal{G} \equiv 8\pi G a^2 \pi^s + \Phi + \Psi - a^2\ddot{E} - 3a\dot{a}E + 2a\dot{B} + 4\dot{a}B = 0$$

where $\delta T^i_i = \delta_{ij}\delta p + \partial_i\partial_j \pi^s$

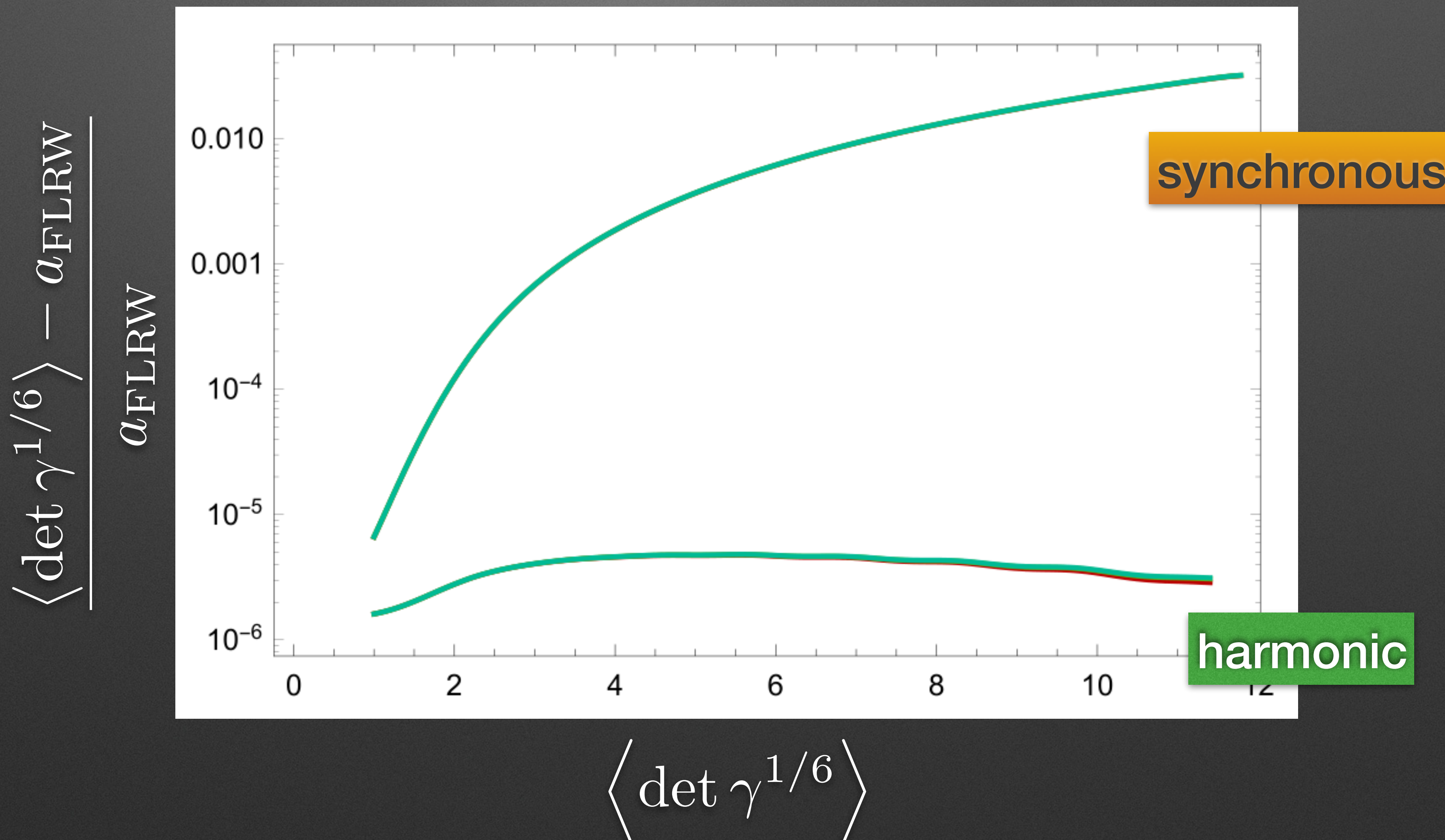
- here we're writing it in terms of the scalar modes only

Violation of the the *linearized* Einstein Equation



$$\mathcal{G} \equiv 8\pi G a^2 \pi^s + \Phi + \Psi - a^2 \ddot{E} - 3a\dot{a}E + 2a\dot{B} + 4\dot{a}B = 0$$

Another example



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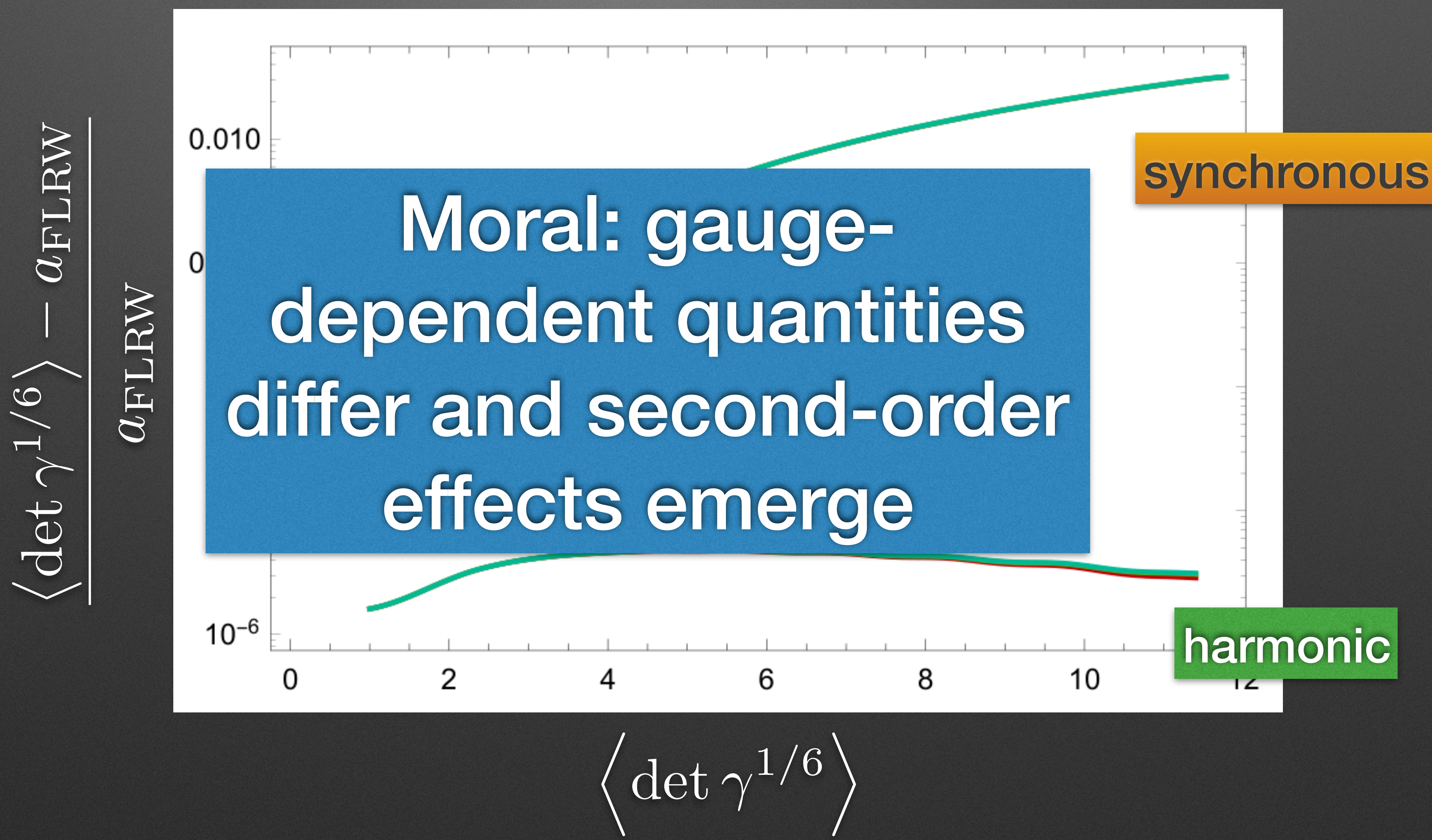

There’s no indication that this has any effect on observables, however.

Your Take-home

- First-Order perturbation theory has a gauge-independent formalism
- Gauge-independent parameters agree well in different gauges/slicing
- **Corrections to these** parameters are gague-dependent and **look** like they change things (but don't yet have observable consequences)

Fin