3 axioms for Maxwell, 3 axioms for Einstein

Q	sources	$p^\mu = (E/c, \vec{p})$
$j^\nu = (c\rho, \vec{j})$	densities	$T_{\mu\nu}$
$A^\mu = (V/c, \vec{A})$	fund. fields	$g_{\mu\nu}$
$\mathcal{D}_2 A = j$	field eq.	$\mathcal{D}_2 g = T$
$(\mathcal{D}_2 A)^\mu := \epsilon_0 c^2 \partial_\nu (\partial^\nu A^\mu - \partial^\mu A^\nu)$	diff. oper.	$(\mathcal{D}_2 g)_{\mu\nu} := \dfrac{c^4}{8\pi G} \left(R_{\mu\nu} - \dfrac{1}{2} R\, g_{\mu\nu} - \Lambda\, g_{\mu\nu}\right)$
$\dot{} := d/d\tau, \quad m\ddot{x}^\mu - qF^\mu{}_\nu \dot{x}^\nu = 0$	test part.	$m\cancel{m}\ddot{x}^\lambda + m\cancel{m}\Gamma^\lambda{}_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$
$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$	auxiliaries	$\Gamma^{\cdot}{}_{\cdot\cdot} = (g^{-1}\partial)^{\cdot} g_{\cdot\cdot}$
$\Delta\varphi = \dfrac{q}{\hbar} \oint A_\mu \dot{x}^\mu$	A-B ; proper time	$\Delta\tau = \oint \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$

8 additive terms	size of $\mathcal{D}_2$	$\sim 10^5$ add. terms in g & g^{-1}
Poincaré, gauge	symmetries	diffeomorphisms
$\partial_\mu j^\mu = 0$	(non-)conservation	$D_\mu T^{\mu\nu} = 0$
linear, 2nd order	uniqueness of $\mathcal{D}_2$	2nd order

$$\epsilon_0 = 8.854187817 \cdot 10^{-12}\,\mathrm{N}^{-1}\mathrm{C}^2\mathrm{m}^{-2}$$

$$\pm\, 0\,\%$$

coupl. const.

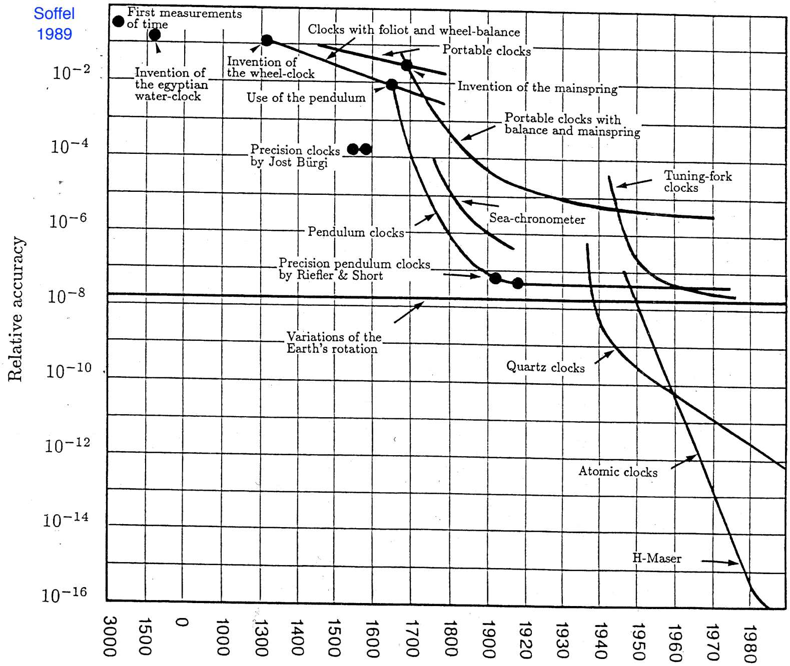
$$G = 6.674208 \cdot 10^{-11}\,\mathrm{N\,kg}^{-2}\mathrm{m}^2$$

$$\pm\, 5 \cdot 10^{-5}$$

$$\Lambda = 1.11 \cdot 10^{-52}\,\mathrm{m}^{-2}$$

$$\pm\, 2\,\%$$

Soffel
1989



First measurements of time
Invention of the egyptian water-clock
Invention of the wheel-clock
Clocks with foliot and wheel-balance
Portable clocks
Invention of the mainspring
Use of the pendulum
Portable clocks with balance and mainspring
Precision clocks by Jost Bürgi
Tuning-fork clocks
Sea-chronometer
Pendulum clocks
Precision pendulum clocks by Riefler & Short
Variations of the Earth's rotation
Quartz clocks
Atomic clocks
H-Maser
Relative accuracy
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
10^{-16}
3000
1500
0
1000
1300
1400
1500
1600
1700
1800
1900
1920
1930
1940
1950
1960
1970
1980

21 octobre 1983: The official death of the meter

The 17th Conférence Générale des Poids et Mesures decides:

"Le mètre est la longueur du trajet parcouru dans le vide par la lumière pendant une durée de $1/299\,792\,458$ de seconde."

distance Earth – Moon $\sim 3.8\ldots \cdot 10^8$ m $\pm\, 1$ cm,

Lunar Laser Ranging from Earth, starting 1962

distance Moon – Earth $\sim 3.8\ldots \cdot 10^8$ m $\pm\, ?$ cm $+$ 30 cm

$A^\mu = (V(r)/c, \vec{0})$ stat. spher. field

$$g_{\mu\nu} = \begin{pmatrix} B(r) & 0 & 0 & 0 \\ 0 & -A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad A, B > 0$$

$V(r) = Q/(4\pi\epsilon_0 r)$ sol. of field eq.

$$B = 1/A = 1 - S/r - \Lambda r^2/3, \;\; S := \frac{2GM}{c^2}$$

Schwarzschild 1916 ($\Lambda = 0$), Kottler 1918

$r = 0$ Coulomb singularities $r = 0$

$$r_{\rm Sch} \sim S + \Lambda S^3/3 \quad \text{Schwarzschild}$$

$$r_{\rm dS} \sim \sqrt{3/\Lambda} - S/2, \;\; \Lambda > 0 \quad \text{de Sitter}$$

$$\frac{d^2r}{dt^2} = \frac{Qq}{m\,4\pi\epsilon_0 r^2}$$

(i) accel. of mass. stat. test part.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}[1 - S/r] + GM\Lambda/3 + c^2\Lambda r[1 - S/r - \Lambda r^2/3]/3$$

(ii) Lensing

Consider a "photon" without spin, $m = 0$, with azimuthal velocity.

Its acceleration with respect to the affine parameter p is purely radial:

$$\frac{d^2 r}{dp^2} = \frac{c^2}{r} - 3\,\frac{MG}{r^2}$$

and *independent* of Λ.

Its acceleration with respect to coordinate time t is also purely radial:

$$\frac{d^2 r}{dt^2} = B\left(\frac{1}{r} - 3\,\frac{MG}{c^2 r^2}\right),$$

but does idepend on Λ via $B = 1 - 2GM/(c^2 r) - \Lambda r^2/3$.

- Does lensing depend on the cosmological constant?

The cosmological principle and kinematics

Let us put $c = 1$. Consider a space-time in coordinates t, r, θ, φ with maximal symmetry on sub-spaces of simultaneity $t =$ const.

$$g_{\mu\nu} \,=\, \mathrm{diag}\left\{1,\, -a(t)^2(1,\, s(r)^2,\, s(r)^2 \sin^2\theta)\right\}$$

with positive 'scale factor' $a(t) > 0$ and

$$s(r) := \begin{cases} \sinh r & \sigma = -1 \\ r & \sigma = \ \ 0 \\ \sin r & \sigma = +1 \quad 0 < r < \pi \end{cases}$$

Its spaces of simultaneity are pseudo-spheres ($\sigma = -1$), Euclidean $\mathbb{R}^3$ ($\sigma = 0$) and spheres ($\sigma = 1$) with radius σa.

Our *point-like* test particles are *(i)* galaxies (or galaxy clusters) with trajectories $t = \tau$, 'cosmic time' and constant r, θ, φ ('comoving') and *(ii)* spinless "photons" travelling on comoving geodesics in 3-space.

Kinematical consequences:

- no lensing, because of maximal symmetry
- horizons, when da/dt is large
- redshift and apparent luminosities.

Redshift: Let a “photon” be emitted in a comoving galaxy at time $t = t_e$, at (comoving) position $r = 0$, with an atomic period T_e and received today $t = t_0$, here $r = r_0$ with atomic period T_0. From the geodesic equation we have:

$$F(t_e, t_0) := \int_{t_e}^{t_0} dt/a(t) = r_0.$$

Pretending that the Milky Way is comoving we also have:

$$F(t_e + T_e, t_0 + T_0) = r_0.$$

Using $T_e \ll t_e$, $T_0 \ll t_0$ to Taylor-expand F to linear order in both arguments, we obtain the spectral deformation

$$z := \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{T_0 - T_e}{T_e} = \frac{a(t_0)}{a(t_e)} - 1.$$

For an expanding universe, $da/dt > 0$ and $z > 0$, ‘redshift’.

Luminosities: Consider a comoving standard candle, e.g. a supernova 1a, emitting many photons isotropically with a known absolute luminosity $L = 10^{35}$ W $\pm 10\,\%$. Let us suppose that all emitted photons reach the 2-sphere around the emission point and containing the Milky Way.

The area of this 2-sphere is:

$$\int_0^\pi d\theta \int_0^{2\pi} d\varphi \sqrt{\det g_{\mu\nu}|_{t_0,r_0}} = 4\pi\, a(t_0)^2 s(r_0)^2,$$

with the restricted metric tensor

$$g_{\mu\nu}|_{t_0,r_0} := \begin{pmatrix} -a(t_0)^2 s(r_0)^2 & 0 \\ 0 & -a(t_0)^2 s(r_0)^2 \sin^2\theta \end{pmatrix}.$$

Therefore the apparent luminosity ℓ, that is the energy received by our detector per unit time and unit surface is:

$$\ell = \frac{L}{4\pi\, a(t_0)^2 s(r_0)^2} \left(\frac{a(t_e)}{a(t_0)}\right)^2.$$

Here one factor of $a(t_e)/a(t_0)$ takes into account that in a say expanding univers, the photons emitted during a time interval T_e arrive in the longer time interval $T_0 = T_e\, a(t_0)/a(t_e)$.

The second factor of $a(t_e)/a(t_0)$ takes into account that each photon arriving in our detector has, by de Broglie's relation, a lower energy $E_0 = \hbar\, 2\pi/T_0$ than at emission.

The Hubble diagram: If we know the scale factor $a(t)$ we can compute the redshift and the apparent luminosity of every standard candle as a function of the emission time t_e for a given fixed arrival time t_0. Unfortunately the arriving photons do not tell us their time of flight. Our parry is eliminating t_e by a parametric plot $(z(t_e), \ell(t_e))$, the famous Hubble diagram. If $a(t)$ is monotonic, say increasing, than we can invert the function $z(t_e)$ and the parametric plot defines a function $\ell(z)$. Then the correspondence $a(t) \to \ell(z)$ is like a Fourier transform and we can ask whether it is invertible. In other words, if we measure the Hubble diagram, can we deduce the scale factor?

Dynamics and Friedman's equations

If we want to apply the cosmological principle to Einstein's equations we must require the same symmetries for the gravitational fields g and for the sources T:

$$T_{\mu\nu} = \mathrm{diag}\left\{(\rho(t),\, p(t)\, a(t)^2(1,\, s(r)^2,\, s(r)^2 \sin^2\theta)\right\}$$

The flat case with constant scale factor teaches us that $\rho(t)$ is an energy density and $p(t)$ a pressure. Now the Einstein equations reduce to two equations,

$$\begin{aligned} 3\left(\frac{a'}{a}\right)^2 + 3\,\frac{\sigma}{a^2} &= 8\pi\, G\rho + \Lambda, \quad ' := \frac{d}{dt}\,, \\ 2\,\frac{a''}{a} + \left(\frac{a'}{a}\right)^2 + \frac{\sigma}{a^2} &= -8\pi\, Gp + \Lambda, \end{aligned}$$

due to Friedman 1922.

Only two equations, but three unknowns: a, ρ, p.
Cheapest parry: put $p = 0$ and call this matter dust or Cold Dark Matter, CDM.
The dust particles are the galaxies.
Then we have a unique local solution with two final conditions, today: $a_0 := a(t_0)$, $H_0 := a'(t_0)/a_0$. Note that in a flat universe, $\sigma = 0$, the initial condition a_0 drops out for arbitrary pressure and is chosen to be one meter or one light year.
The following dimensionless parameters are used:

$$\Omega_m := \frac{8\pi G\rho_0}{3H_0^2}, \quad \Omega_\Lambda := \frac{\Lambda}{3H_0^2}, \quad \Omega_k := -\frac{\sigma}{H_0^2 a_0^2},$$

with the mass density today $\rho_0 := \rho(t_0)$. From Friedman's first equation we have: $\Omega_m + \Omega_\Lambda + \Omega_k = 1$.

Best fit: ΛCDM

Super Novae + Baryonic Acoustic Oscillations + Planck (2015):

$$\Omega_k = 0.0008 \pm 0.0020.$$

analytic solutions for *flat* universes filled with dust only:

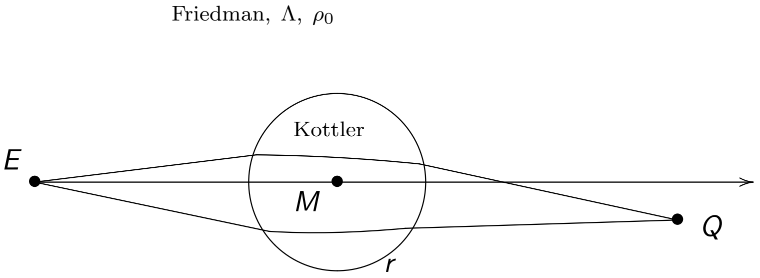
$$\begin{aligned} a(t) &= a_0 \left(\frac{8\pi G\rho_0}{\Lambda}\right)^{1/3} \sinh^{2/3}\left[\tfrac{1}{2}\sqrt{3\Lambda}\, t\right], \\ \rho(t) &= \rho_0 \left(\frac{a_0}{a(t)}\right)^3, \end{aligned}$$

$$\Omega_\Lambda = 0.690 \pm 0.006, \quad H_0^{-1} = 14.42\,\mathrm{Gy} \pm 1\%.$$

big bang: $t = 0$, age of universe:

$$t_0 = \operatorname{arcosh}\frac{1+\Omega_\Lambda}{1-\Omega_\Lambda}\ \frac{1}{3\sqrt{\Omega_\Lambda}}\, H_0^{-1} = 13.78\,\mathrm{Gy}.$$

The Einstein-Straus solution 1945



$$\rho_0 = M/(\tfrac{4}{3}\pi r^3)$$